Part 1: Answer two of the five questions given below.

Ph.D students: At most one question can be of the M.A. type.

Only two questions will be graded. Write your chosen question numbers here:

(M.A.) 1. Characterize the type of the critical points as a function of $\alpha$ for the system,

\[
\frac{dx}{dt} = (1-x^2)y, \\
\frac{dy}{dt} = -x - \alpha y.
\]

(M.A.) 2. Construct the total energy function, find and classify the equilibrium points, derive and solve the corresponding linear equations, and sketch a global phase plane portrait.

\[
\frac{d^2x}{dt^2} + x + \frac{3}{x - 4} = 0.
\]

3. (a) Show that if $(x_0, y_0)$ is a local minimum of $V(x, y)$, then $F(x, y) = V(x, y) - V(x_0, y_0)$ is a Liapunov function for the gradient flow,

\[
\frac{dx}{dt} = -\frac{\partial V(x, y)}{\partial x}, \\
\frac{dy}{dt} = -\frac{\partial V(x, y)}{\partial y},
\]

and that this guarantees stability of $(x_0, y_0)$.

(b) Using the Liapunov stability theorem [State it.], can you determine if the origin is asymptotically stable or only that it is stable for the gradient flow,

\[
\frac{dx}{dt} = 4xy - 4x^3, \\
\frac{dy}{dt} = 2x^2 - 2y,
\]
4. Let $A$ be a $2 \times 2$ matrix with scalar entries. Then (by Cayley-Hamilton) $A$ satisfies its characteristic equation, i.e. if $p(\lambda) = \text{det}(A - \lambda I)$, then $p(A) = 0$.

(a) Show that $Le^{tA} = 0$, where the differential operator $L$ is defined by $L = p\left(\frac{d}{dt}\right)$.

(b) Assume that the following is valid [It is].

\[
e^{tA} = v(t)I + w(t)A,
\]
\[
\frac{d}{dt}e^{tA} = \frac{d}{dt}v(t)I + \frac{d}{dt}w(t)A.
\]

Show that $Lv(t) = 0$, and $Lw(t) = 0$. Find $v(0), w(0), \frac{d}{dt}v(0), \frac{d}{dt}w(0)$.

(c) Use this information to compute $e^{tA}$, where

\[
A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.
\]

5. Consider the system

\[
\frac{dx}{dt} = x(y - y^2),
\]
\[
\frac{dy}{dt} = x^2 - y.
\]

(a) Find the critical points and determine their stability.

(b) Show that if $x(0) > 0$, then $x(t) > 0$ for all $t \geq 0$.

(c) What is $\lim_{t \to \infty} (x(t), y(t))$ if $x(0) \geq 0$?

(Either sketch the trajectories of the phase plane portrait or argue based on your previous work.)

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→ Part 2: Answer two of the five questions given below.

Ph.D students: At most one question can be of the M.A. type.
Only two questions will be graded. Write your chosen question numbers here:

(M.A.) 1(a). Find the earliest time at which a singularity develops for the equation
\[ u_t + uu_x = 0, \quad -\infty < x < \infty, \quad 0 < t, \quad u(x, 0) = \sin x. \]

(M.A.) 1(b). Find the solution and domain of existence for
\[ xu_x + (x + y)u_y = 1, \]
with the boundary condition, \( u(1, y) = y, \quad 0 < y < 1. \)

(M.A.) 2. Find the solution to the wave equation,
\[ y_{tt} = c^2 y_{xx}, \quad 0 < t, \quad 0 < x, \]
that satisfies the boundary condition, \( y(0, t) = s(t), \) and the initial conditions, \( y(x, 0) = 0, \quad y_t(x, 0) = g(x). \)

Hint: Try \( y(x, t) = f(x - ct) + h(x + ct). \)

3. Let \( \Omega = G \times (0, T), \) where \( G \) is a bounded smooth domain in \( \mathbb{R}^3. \) Show that there is a
unique solution to
\[ U_t = \nabla \cdot (\kappa(\underline{z}) \nabla U) - c(\underline{z}) U, \quad (\underline{z}, t) \in \Omega \]
with \( U = f \) on \( \partial \Omega. \) The functions \( f, \kappa, c \) are smooth and \( c \) is positive.

4(a). Show that a fundamental solution of the differential operator, \( \Delta + \frac{\omega^2}{c^2}, \Delta, \) the Laplacian
in \( \mathbb{R}^3, \) is
\[ G(\underline{z}, \zeta) = -\frac{e^{-i\frac{\omega}{c^2} r}}{4\pi r}, \]
where \( r = | \underline{z} - \zeta |. \)

4(b). Using (a), find the solution to
\[ (\Delta + \frac{\omega^2}{c^2})U(\underline{z}) = f(\underline{z}), \]
in a smooth bounded region \( \Omega \) with \( U = 0 \) on \( \partial \Omega. \)

5. Prove: If the functions \( w_n \) are harmonic in a bounded domain \( G \) in \( \mathbb{R}^2, \) continuous in \( \overline{G}, \) then the sequence \( \{w_n\} \) is uniformly convergent throughout \( \overline{G}, \) and the limit function is harmonic in \( G. \)