August 1994 ODE/PDE EXAM
ID NUMBER:

Part 1. Solve two of the four problems given below.
Ph.D students: At most one question can be of the M.A. type.
Only two problems will be graded. Write your problem numbers here:

1. (M.A.) Consider the system \( x' = Ax + f(t) \) with \( x(t) \), \( f(t) \) in \( \mathbb{R}^n \), and \( f(t) = f(t + T) \). All of the eigenvalues of the constant \( n \times n \) matrix \( A \) satisfy the constraint, \( \Re(\lambda) < 0 \).
   a. Write down a formula for a general solution.
   b. Write down a formula for a \( T \)-periodic solution \( h(t) \) and show that it is unique.
   c. Show that \( \lim_{t \to -\infty} x(t) = h(t) \).

2. (M.A.) Let \( A \) be an arbitrary \( 2 \times 2 \) matrix with real entries. Then (by Cayley-Hamilton) \( A \) satisfies its characteristic equation, i.e., if \( p(\lambda) = \det(A - \lambda I) \), then \( p(A) = 0 \).
   a. Show that \( Le^{tA} = 0 \), where the differential operator \( L \) is defined by \( L = p\left( \frac{d}{dt} \right) \).
   b. Assume that the following is valid. (The functions \( v(t) \) and \( w(t) \) are real-valued.)

\[
\begin{align*}
   e^{tA} & = v(t)I + w(t)A, \\
   \frac{d}{dt} e^{tA} & = \frac{dv}{dt}(t)I + \frac{dw}{dt}(t)A.
\end{align*}
\]

Show that \( L v(t) = 0 \) and \( L w(t) = 0 \). Find \( v(0) \), \( w(0) \) \( v'(0) \) \( w'(0) \).
   c. Use this information to construct \( e^{tA} \) where

\[
A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}.
\]
3. Show that

\[ \frac{dx}{dt} = F(x, y) = -2x - 3y - xy^2, \]
\[ \frac{dy}{dt} = G(x, y) = y + x^2 - x^2y, \]

has no periodic solutions that are not constant.

Hint: Use Green's Theorem to study

\[ \oint_C [F\,dy - G\,dx] \]

on a smooth closed curve \( C \).

4.

a. Show that

\[ \mathcal{H} = \frac{1}{2}y^2 + \frac{1}{2}v^2 - \frac{1}{2}\lambda^2 x^2 - \frac{1}{2}\mu^2 u^2 + \frac{1}{2}ux^2 \]

is a Hamiltonian for

\[ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = \lambda^2 x - ux, \]
\[ \frac{du}{dt} = v, \quad \frac{dv}{dt} = \mu^2 u - \frac{1}{2}x^2. \]

b. Show that for \( \lambda^2 = \mu^2 \), the above system can be reduced to a two-dimension system. \{Hint: Set \( u = \alpha x, \ v = \alpha y \) with a suitable \( \alpha \).\}

c. Find a Hamiltonian for the reduced system.

d. Discuss the critical points and sketch a phase plane portrait of the reduced system.
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Part 2. Solve two of the four problems given below.

Ph.D students: At most one question can be of the M.A. type.

Only two problems will be graded. Write your problem numbers here:

1. (M.A.)
   a. Classify the PDE,
      \[ u_{xx} + \alpha u_{yy} + \beta u_y = \lambda^2 u, \]
      for all values of \( \alpha \) and \( \beta \).
   b. Solve for \( u(x, y) \) with \( \alpha = \beta = \lambda^2 = 0 \) on \( 0 < x < 1 \).
      Is there a maximum principle in this case? Justify your answer with the solution you obtained.
   c. Set \( \beta = 0 \) and \( \alpha = -c^2 \). Solve for
      \[ u = f(s) = \sum_{n=0}^{\infty} a_n s^n , \]
      where \( s = y^2 - c^2 x^2 \). (First find an equation for \( f(y^2 - c^2 x^2) \).)

2. (M.A.) State and prove a theorem regarding the attainment of maxima and minima by solutions to the I.B.V.P.

   \[ u_t = u_{xx}, \quad 0 < x < L, \quad t > 0, \]
   \[ u(t, 0) = u(t, L) = 0, \quad t > 0, \]
   \[ u(0, x) = f(x) \in C^2(0, L). \]

3. 
   a. Show that
      \[ u(x, y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2} \]
      is harmonic for \( x^2 + y^2 < 1 \).
b. Is it harmonic for \( x^2 + y^2 < a \) if \( a > 1 \)? Explain.

c. Prove that
\[
\frac{a-r}{a+r} \leq u(x,y) \leq \frac{a+r}{a-r}
\]
for \( r < a \) is true for all \( a < 1 \), where \( r = \sqrt{x^2 + y^2} \).

d. Prove that
\[
\frac{1-a}{1+a} \leq u(x,y) \leq \frac{1+a}{1-a}
\]
for \( \sqrt{x^2 + y^2} \leq a \) is true for all \( a < 1 \).

4. Consider the following equation that arises in the study of elasticity,
\[
\rho u_{tt} = (\lambda + 2\mu) \nabla (\nabla \cdot u) - \mu \nabla \times (\nabla \times u) + F(x); \quad u = (u_1, u_2, u_3)
\]
in \( \Omega \subset \mathbb{R}^3 \) with \( u = 0 \) on the boundary of \( \Omega \). (\( \rho, \lambda, \mu \) are constants.)

a. Show that if \( F = 0 \), then
\[
E(t) = \int_{\Omega} \frac{1}{2} [\rho |u_i|^2 + (\lambda + \mu)(\nabla \cdot u)^2 + \mu \sum_{i=1}^{3} |\nabla u_i|^2] \, dx
\]
is a constant.

b. Assume \( F \neq 0 \), and set \( F = \nabla U + \nabla \times \theta \), and \( u = \nabla \phi + \nabla \times A \). Find the equations for \( \phi \) and \( A \).

c. Taking \( \phi = \phi_0, \ A = A_0 \) (constants) on \( \partial \Omega \), find the Lagrangian for the \( (\phi, A) \) equations in b.