1. Consider the subspace $M \subset \mathbb{R}^4$ spanned by the vectors 
\[ v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}. \]
(a) Write down the matrix $P$ representing the orthogonal projector onto $M$.
(b) What is the null space of $P$?
(c) What is the range of $P$?
(d) Find the vector $x$ in $M$ that is closest in the 2-norm sense to the vector $c = (1, 1, 1)^T$.

2. Consider solving the linear system of equations, $Ax = b$ with $A$ an $n \times n$ real matrix and $x, b \in \mathbb{R}^n$, using an iterative method based on the splitting $A = M - N$, with $M$ invertible. The iterative method is $x_{k+1} = Gx_k + f$ where the iteration matrix is $G = M^{-1}N$ and $f = M^{-1}b$.
(a) What are $G_J$ and $G_{GSS}$, the iteration matrices for the Jacobi and Gauss-Seidel iterative methods, respectively?
(b) The iterative method is called symmetrizable if for some nonsingular matrix $W \in \mathbb{R}^{n \times n}$, the matrix $W(I - G)W^{-1}$ is symmetric positive definite. $W$ is called the symmetrization matrix. If the iterative method is symmetrizable then prove:
   i. The eigenvalues of $G$ are real.
   ii. The largest eigenvalue of $G$ is less than one.
   iii. The set of eigenvectors of $G$ forms a basis for $\mathbb{R}^n$.
(c) Does a symmetrizable iterative method necessarily converge? Explain.
(d) Show that the iterative method is symmetrizable whenever $A$ and $M$ are symmetric positive definite matrices. (Hint: consider $A^{1/2}$.)
(e) If $A$ is symmetric positive definite, are Jacobi or Gauss-Seidel symmetrizable? Explain.
3. Let: \[ A = U\Sigma V^T, \]
be the singular value decomposition of the \( m \times n \) matrix \( A \).

i. Supposing \( \text{rank}(A) = r \), use \( U \) and \( V \) to find orthonormal bases for the column space of \( A \), the row space of \( A \), the null space of \( A \), and the null space of \( A^T \).

ii. Supposing \( m = n \), use the SVD to find the Euclidean norm condition number of \( A \). Justify your answer.

iii. Use the SVD to find the matrix, \( B \), of rank \( q < r \) which is closest to \( A \) in the Euclidean norm. Justify your answer.

iv. Let \( W \) be an \( n \times n \) matrix. Suppose the singular values of \( W \) are:

\[ \sigma_k = 2^{1-k}. \]

Using standard double precision arithmetic, for which \( \epsilon_{\text{machine}} \approx 2^{-52} \), for \( n \) how large can one expect to accurately solve a linear system with coefficient matrix \( W \)?

v. Consider the problem of computing \( Wz \) using standard double precision arithmetic and \( W \) from the previous problem. Assuming you have precomputed the SVD, roughly how many flops should this require for \( n \gg 1 \)? (Compute the approximate leading constant for full credit.)

4. Suppose \( A = A^T \) is symmetric but not necessarily positive definite. Assume further that \( A \) has an \( LU \) factorization. Describe an algorithm for factoring \( A \) into triangular factors requiring about half the work and storage of standard Gaussian elimination. Comment on the stability of the algorithm, justifying your comments.