1. a. Let $A$ and $B$ be real $m \times n$ matrices. What can you say, in general, about the relationship of the three numbers $\text{rank}(A)$, $\text{rank}(B)$, $\text{rank}(A + B)$?

b. Let $A$ be a real $m \times n$ matrix. What can you say, in general, about the relationship of the three numbers $\text{rank}(A)$, $\text{rank}(A^T A)$, $\text{rank}(A A^T)$?

Prove your claims.

2. a. Let $A$ be a real $m \times n$ matrix. Define the singular value decomposition of $A$.

b. Suppose $A$ is a $7 \times 10$ matrix with singular values

$$\sigma_1 = 10, \quad \sigma_2 = 5, \quad \sigma_3 = 1,$$

and all other singular values of $A$ are zero.

What is the rank of $A$? What is the distance of $A$ to the nearest matrix of rank 2? What is the distance of $A$ to the zero matrix? Explain which distance function you use and justify your answers.

3. Given $n$ real numbers $r_j$, $j = 1, \ldots, n$, we define an $n \times n$ circulant matrix, $C$, by:

$$c_{ij} = r_{j-i+1}, \quad r_p = r_{p+n}, \quad p \leq 0.$$ 

That is:

$$C = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ r_n & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_2 & r_3 & \cdots & r_1 \end{pmatrix}.$$

a. Show that the discrete Fourier vectors, $q^{(k)}$, defined by:

$$q^{(k)}_j = e^{2\pi i (j-1)(k-1)/n}, \quad j, k = 1, \ldots, n,$$

are eigenvectors of $C$. What are the corresponding eigenvalues?

b. Sketch a fast algorithm for computing $C v$ and $C^{-1} v$ for favorable values of $n$. What is the approximate operation count of this fast algorithm?

Let:

$$B = C + uv^T,$$

where $u, v \in \mathbb{R}^n$. Describe a fast algorithm for solving $Bx = y$.

4. Describe three basic algorithms for finding the least squares solution of an overdetermined system of $m$ linear equations in $n$ unknowns. How do they compare in terms of cost? Discuss their accuracy.