Instructions: Complete all four problems.

1. Consider the problem of interpolating two-dimensional data: \((x_i, y_i, f(x_i, y_i)), \ i = 1, \ldots, n,\) by a linear function, \(l(x, y) = ax + by + c.\) That is, consider the problem of determining the constants \(a, b, c\) so that:

\[
l(x_i, y_i) = f(x_i, y_i), \ i = 1, \ldots, n.
\]

(i) What is the coefficient matrix of the system?

(ii) Give detailed necessary and sufficient conditions for the existence of a unique solution for arbitrary data, \(f.\) Give a geometrical interpretation of your result.

(iii) How does your result generalize to three space dimensions?

2. Gerschgorin’s Circle Theorem states:

Let \(A\) be an \(n \times n\) matrix, and let \(D_i,\) the \(i\)th Gerschgorin disk, be defined as the set of complex numbers, \(z,\) satisfying:

\[
|z - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}|.
\]

Then if \(\lambda\) is an eigenvalue of \(A,\)

\[
\lambda \in \bigcup_{i=1}^{n} D_i
\]

(i) Prove Gerschgorin’s Circle Theorem.

(ii) Prove in addition that if \(D_i\) is an isolated Gerschgorin disk, that is if \(D_i\) does not intersect any of the other disks, then it contains exactly one eigenvalue.
3. Let $A$ and $B$ be be $2n \times 2n$ matrices. Write them in block form as:

$$
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}, \quad B = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}.
$$

where the submatrices $A_{ij}$, $B_{ij}$, are $n \times n$.

(i) Write the product $AB$ in block form using the block forms of $A$ and $B$.

(ii) Define the matrices:

$$
P_1 = (A_{11} + A_{22})(B_{11} + B_{22}), \quad P_2 = (A_{21} + A_{22})B_{11},
$$

$$
P_3 = A_{11}(B_{12} - B_{22}), \quad P_4 = A_{22}(B_{21} - B_{11}), \quad P_5 = (A_{11} + A_{12})B_{22},
$$

$$
P_6 = (A_{21} - A_{11})(B_{11} + B_{12}), \quad P_7 = (A_{12} - A_{22})(B_{21} + B_{22}).
$$

Let $C_{ij}$ denote the blocks of $AB$. Show how to express $C_{11}$ as a linear combination of the matrices $P_j$. (The other blocks can be similarly expressed.)

(iii) Make a detailed flop count for both the conventional matrix multiplication algorithm and a block algorithm based on computing the $P_j$'s. Which is more efficient? (You need only count multiplications.) Suggest a method for significantly improving the efficiency of matrix multiplication for special values of the matrix order.

4. Let $A$ be an $n \times n$ matrix.

(i) Define the condition number of $A$.

(ii) Show that, for any subordinate matrix norm, and any vector $x$,

$$
\|A^{-1}\| \leq \frac{\|x\|}{\|Ax\|}.
$$

(Recall that the matrix norm subordinate to a vector norm is defined by:

$$
\|A\| = \sup_{x \neq 0} \|Ax\|/\|x\|.
$$

(iii) Describe how, when using the Gaussian elimination algorithm, one might practically estimate the condition number, using the maximum norm. By a practical estimate we mean one which does not increase the dominant cost of the solution process.)