Instruction: Complete all four problems.

1. In this problem all matrices are real and of dimension \( n \times n \).
   a) Define a normal matrix.
   b) Define an orthogonal matrix.
   c) State the properties of the eigenvalues and eigenvectors of a normal matrix? Prove your statements.
   d) Let \( Q \) denote an orthogonal matrix and let \( \Lambda \) denote a diagonal matrix. Is the matrix \( A = Q^{-1} \Lambda Q \) always normal? (Give a proof or counterexample.)

2. Consider an overdetermined linear system
   \[
   Ax = b
   \]
   where \( A \) is real of size \( m \times n \), \( x \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \), \( m > n \).
   a) When is a vector \( x^* \in \mathbb{R}^n \) called a least squares solution of the system \( Ax = b \)?
   b) Give assumptions on \( A \) so that a least squares solution exists and is uniquely determined. Prove the result.
   c) Compute the least squares solution of
   \[
   \begin{pmatrix}
   1 & 0 \\
   1 & 1 \\
   0 & 1
   \end{pmatrix}
   \begin{pmatrix}
   x_1 \\
   x_2
   \end{pmatrix}
   =
   \begin{pmatrix}
   1 \\
   2 \\
   3
   \end{pmatrix}.
   \]

3. Let \( A \) be an \( n \times n \) positive definite symmetric matrix.
   a) Prove that \( A = LL^T \) where \( L \) is lower triangular.
   b) Estimate, as a function of \( n \), the number of arithmetic operations required to compute \( L \) by elimination.
   c) Using the matrix norm subordinate to the standard Euclidean vector norm, relate \( \| L \| \) to \( \| A \| \). Does this have any implications for the numerical stability of the computation of \( L \)?

4. Suppose \( A \) is an \( n \times n \) matrix with a known \( LU \) factorization and let \( B \) be a rank-one perturbation of \( A \), i.e.
   \[
   B = A + uv^T.
   \]
   Assuming both \( A \) and \( B \) are invertible, describe how to compute the solution to:
   \[
   Bx = c
   \]
   in \( O(n^2) \) operations.