NUMERICAL ANALYSIS QUALIFYING EXAM

Each Problem Counts 25 Points

January 1994

Name: ______________________

1. (i) State the fundamental theorem of linear algebra.
   (ii) Find the $LU$ decomposition of the following matrix $A$.
   (iii) Find bases for each of the four fundamental subspaces associated with $A$ (that is, $\mathcal{R}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A)$, and $\mathcal{N}(A^T)$), and state the dimension of these subspaces.

   $$A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

2. (i) Describe the singular value decomposition of an $m \times n$ matrix $A$. Define all matrices that you introduce.
   (ii) For the following matrix $A$ and vector $b$, find the singular value decomposition, the pseudoinverse $A^+$, and the minimum length least squares solution $x^+$ of $Ax = b$.

   $$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}.$$

3. A real square matrix $A$ is called positive definite symmetric (PDS) if it is symmetric, $A = A^T$, and, for any $x \neq 0$:

   $$x^T Ax > 0.$$

   (i) Show that all eigenvalues of a PDS matrix are real and positive.
   (ii) Let $A^{(m)}$ be the $m \times m$ matrix obtained by intersecting the first $m$ rows and columns of a PDS matrix, $A$. Show that $A^{(m)}$ is also PDS.
   (iii) Use the results above to show that if Gaussian elimination without pivoting is applied to a PDS matrix, only positive pivots are encountered. (Hint: consider the relationship between the pivots, the determinant, and the eigenvalues.)
   (iv) Use (iii) to prove the existence of the Cholesky decomposition of a PDS matrix: $A = LL^T$ where $L$ is lower triangular.
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4. The power method for computing an eigenvalue of a matrix, $A$, is defined by:

$$y_{n+1} = Ax_n, \quad \lambda_{n+1} = y_{n+1}^T x_n, \quad x_{n+1} = \frac{y_{n+1}}{\sqrt{y_{n+1}^T y_{n+1}}}.$$  

where $x_0$ satisfying $x_0^T x_0 = 1$ is otherwise arbitrary.

(i) Show that if there is a single extreme eigenvalue, that is a simple eigenvalue $\lambda$ such that $|\alpha| < |\lambda|$ for all other eigenvalues $\alpha$, then the power method converges, that is $\lambda_n \to \lambda$, for most $x_0$. (To simplify your arguments, assume that $A$ is diagonalizable.)

(ii) Describe the typical behavior of the method if the extreme eigenvalues correspond to a conjugate imaginary pair. In particular show that the sequence $\lambda_n$ may converge to a number which is not an eigenvalue. (An example will do.)

(iii) Let $A$ be a real skew-symmetric matrix, i.e. $A = -A^T$. Show that all eigenvalues of $A$ are imaginary. What can you say about the eigenvalues of $A^2$?

(iv) Suggest a modification of the power method to compute extreme eigenvalues of a skew-symmetric matrix.