Complex Analysis Qualifying Exam

August 1999

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let \( f \) be analytic on \( \mathbb{C} \) and real-valued on the circle \( |z| = 1 \). Show that \( f \) is a constant on \( \mathbb{C} \).

2. Classify the singularities at \( z = 0 \) of the following functions \( f(z) \) (including the point at \( \infty \)).

   a) \( f(z) = \frac{\sin^2 z}{z^4} \).

   b) \( f(z) = \sin \left( \frac{1}{z} \right) + \frac{1}{z^2(z - 1)} \).

   c) \( f(z) = \csc z - \frac{1}{z} \).

3. Show the map \( f(z) = \frac{z - i}{z + i} \) is a bijection of the upper half plane \( H = \{ z \in \mathbb{C} | \text{Im } z > 0 \} \) onto the unit disc \( D = \{ z \in \mathbb{C} : |z| < 1 \} \).

4. Let \( f \) be a continuous map of a connected open subset \( U \subset \mathbb{C} \) into \( \mathbb{C} \).

   a) Show that \( f \) has a primitive on \( U \) if and only if \( \int_{\gamma} f(z)dz = 0 \) for every simple closed curve \( \gamma \) contained in \( U \).

   b) Consider \( f(z) = \frac{1}{z} \) on the punctured unit disc \( D(0, 1) - \{0\} \). Does \( f \) have a primitive on \( D(0, 1) - \{0\} \)? Explain.

5. Use the theory of residues to evaluate the integral \( \int_{0}^{\infty} \frac{\ln x \, dx}{x^2 + a^2} \).

6. State the Argument Principle and use it to prove the Open Mapping Theorem: Let \( f \) be analytic on some region \( \Omega \). Then the image \( f(U) \) is open in \( \mathbb{C} \) for every open set \( U \subset \Omega \). Hint. Apply the Argument Principle to the function \( f(z) - w \).

7. Use the Casorati-Weierstrass Theorem to prove that if the composition \( f \circ g \) of two entire functions \( f \) and \( g \) is a polynomial, then both \( f \) and \( g \) are polynomials.

8. Let \( f \) be a bounded analytic function on the disc \( D(0, R) \). Suppose that \( f \) also satisfies \( f^{(i)}(0) = 0 \) for all \( i = 0, \cdots, k \). Show that \( f \) satisfies the inequality \( |f(z)| \leq \frac{M}{R^{k+1}} |z|^{k+1} \) on \( D(0, R) \) where \( M = \sup_{z \in D(0, R)} |f(z)| \).