Complex Variables Master's Qualifying Examination
January 1996

1) (a) Determine the region of absolute convergence of the products:
   (i) \( \prod_{n=1}^{\infty} (1 - z^n) \)
   (ii) \( \prod_{n=1}^{\infty} (1 + z^{2n}) \)
   (iii) \( \prod_{n=0}^{\infty} (1 + c_n z) \) where \( \sum_{n=0}^{\infty} |c_n| < \infty \)
   (iv) \( \prod_{n=1}^{\infty} \left( 1 - \frac{1}{n^2} \right) \)

(b) For the following functions in the extended complex plane give the branch points and the isolated singular points. Say whether the isolated singular points are removable, essential or poles (and, if poles, give the order)
   (a) \( \sqrt{z^2 - 1} \)
   (b) \( \exp \left( \frac{1}{\sin z} \right) \)
   (c) \( \ln(\sqrt{z^2 + 1}) \)
   (d) \( \tan z \)

2) Let \( D = \{ z \in \mathbb{C} \mid |z| \leq 1 \text{ or } |z - 2| \leq 1 \} \). Let \( \tilde{D} = (\mathbb{C} \setminus D) \cup \{ \infty \} \). Show \( \tilde{D} \) is conformally equivalent to the open upper half plane.

3) (a) Let \( f \) and \( g \) be entire functions satisfying \( |f(z)| \leq |g(z)| \) for \( |z| \geq 100 \). Assume \( g \) is not identically zero. Show \( f/g \) is rational.

(b) Let \( u \) be harmonic in \( \mathbb{C} \) and \( u(x, y) \geq -2 \) for all \( x + iy \in \mathbb{C} \). Show \( u \) is constant in \( \mathbb{C} \).

4) Let \( f(z) = z^4 - 5z + 1 \)
   (a) How many zeros does \( f(z) \) have in the disc \( \{ z \in \mathbb{C} \mid |z| < 1 \} \).
   (b) How many zeros does \( f(z) \) have in the annulus \( 1 < |z| < 2 \).

5) Compute \( F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + 1} \, dx \) where \( k \in \mathbb{R} \)

6) Expand (if possible) in Laurent series in the indicated region
   (a) \( e^{1/(z-1)} \mid z \mid > 1 \)
   (b) \( \frac{1}{(z-a)(z-b)} \)
      (i) \( 0 < |a| < |z| < |b| \)
      (ii) \( |a| < |b| < |z| \)
(c) \( \log \left( \frac{1}{1 - z} \right) \quad \text{if } |z| > 1 \)

7) Evaluate the integral

\[
\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} \, dx
\]

8) Choose a branch of \( \sqrt{z^2 - 1} \) that is analytic on \( \mathbb{C}\backslash\{x + 0i \mid -1 \leq x \leq 1\} \) and has the value \( \sqrt{3} \) at \( z = 2 \). Evaluate \( \int_{\gamma} \sqrt{z^2 - 1} \, dz \) where \( \gamma \) is the circle of radius 2, centered at 0 and oriented counterclockwise.