1. Let \( \{a_n\} \) be a sequence of complex numbers. Assume that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) converges for all \( z \) satisfying \( |z| \leq r \). Prove that if \( |a_1| > \sum_{n=2}^{\infty} n|a_n|r^{n-1} \), then \( f \) is an injective function on the disc \( |z| \leq r \).

2. Let \( f(z) = \frac{1}{z-1} + \frac{1}{(z-2)^2} \). Expand \( f(z) \) in a
   i) Taylor series in \( |z| < 1 \)
   ii) Laurent series in \( 1 < |z| < 2 \).

3. A complex-valued function \( f = U + iV \) is said to be harmonic on a domain \( D \subset \mathbb{C} \) if \( U \) and \( V \) are harmonic on \( D \). Show that \( f \) is holomorphic on \( D \) if and only if both \( f \) and \( zf \) are harmonic on \( D \).

4. (a) Show \( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \) is meromorphic in the complex plane \( \mathbb{C} \)
   (b) Argue that \( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \) is holomorphic in \( \mathbb{C} \)
   (c) Show that this holomorphic function is 0, i.e. \( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=-\infty}^{\infty} \frac{z^2}{(\sin \pi z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \)
5. Let \( f : U \to \mathbb{C} \) be a holomorphic function defined on an open subset \( U \) of \( \mathbb{C} \). Let \( R \) be a closed rectangle contained in \( U \) (assume that the sides of \( R \) are parallel to the lines \( \text{Re}z = 0 \) and \( \text{Im}z = 0 \)). Give a complete proof of the equality

\[
\int_{\partial R} f(z)\,dz = 0
\]

6. Let \( p_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} \). Prove that for every \( R > 0 \) there exists a positive integer \( n(R) \) such that all roots of \( p_n(z) = 0 \) for \( n \geq n(R) \) belong to the set \( \{z \in \mathbb{C} \mid |z| > R\} \).

7. Assume that \( a, b, c \) are real numbers satisfying \( ac - b^2 > 0 \). Prove using residues

\[
\int_{-\infty}^{\infty} \frac{dx}{ax^2 + 2bx + c} = \frac{\pi}{\sqrt{ac - b^2}}.
\]

8. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of complex numbers. Assume that \( \{a_n\} \) has no accumulation point. Prove that there exists a holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) such that \( f(a_n) = b_n \).