Complex Variables
Master's Examination

Spring 2000

Instructions: There are nine (9) questions on this examination, and each question is worth 25 points. Work any 8 problems. A maximum score of 200 points is possible.

1. Find a conformal mapping of the strip $0 < \Re z < 1$ onto the unit disk in such a way that $z = 1/2$ goes to $w = 0$ and $z = \infty$ goes to $w = 1$.

2. According to the Weierstrass factorization theorem, $f(z) = \cos \sqrt{z}$ can be written as an infinite product

   $$f(z) = C e^{g(z)} z^m \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) e^{h_n(z)} ,$$

   where $g(z)$ is entire and

   $$h_n = \begin{cases} 0, & k = 0 \\ \frac{1}{a_n} + \frac{1}{2} \left( \frac{z}{a_n} \right)^2 + \cdots + \frac{1}{k} \left( \frac{z}{a_n} \right)^k , & k \in \mathbb{Z}^+ \end{cases}$$

   Determine $a_n$, $k$ and $m$.

3. According to the Mittag-Leffler theorem, the meromorphic function

   $$f(z) = \frac{\pi^2}{\sin^2 \pi z} ,$$

   can be expressed by the infinite series

   $$f(z) = \sum_{k} \left[ P_k \left( \frac{1}{z - b_k} \right) - p_k(z) \right] + g(z) ,$$

   where $P_k(z)$, $p_k(z)$ are appropriately chosen polynomials, $b_k$ are appropriately chosen complex numbers and $g(z)$ is analytic in the entire complex plane. Determine $b_k$, $P_k$ and $p_k$.

4. For $a, b > 0$, evaluate the integral

   $$\int_{0}^{\infty} \frac{\cos ax}{x^2 + b^2} dx .$$

   Carefully justify any estimate you make.

5. Evaluate the integral

   $$\int_{0}^{\infty} \frac{x^p}{1 + x^2} dx ,$$

   with $-1 < p < 1$ by contour integration. As in the previous problem, carefully justify all your estimates.
6. Let \( B(0; 1) = \{ z \in \mathbb{C} \mid |z| < 1 \} \) be the unit disk. If \( a > e \) and \( n \) is a positive integer, prove that the equation \( e^z = az^n \) has \( n \) distinct roots in \( B(0; 1) \) (counted with multiplicity).

7. Let \( \Omega \subset \mathbb{C} \) be a simply connected region, and \( u : \Omega \to \mathbb{R} \) be a harmonic function. Prove that there exists \( v : \Omega \to \mathbb{R} \) such that \( u + iv \) is analytic on \( \Omega \).
   \textit{Hint:} Consider \( g(z) = \frac{\partial u}{\partial x} + i \left( -\frac{\partial u}{\partial y} \right) \).

8. Assume that \( f(z) \) is analytic on \( \mathbb{C} \setminus \{0\} \) and
   \[ |f(z)| \leq |z|^2 + \frac{1}{|z|^{1/2}} \]
   for all \( z \in \mathbb{C} \setminus \{0\} \). Prove that \( f \) is a polynomial of degree at most 2.

9. Let \( f(z) \) be continuous on the closed right half-plane \( \tilde{\mathcal{H}} = \{ z \in \mathbb{C} \mid \Re z \geq 0 \} \) and analytic on the open right half-plane \( \mathcal{H} = \{ z \in \mathbb{C} \mid \Re z > 0 \} \). Suppose there exist constants \( C, M \in \mathbb{R} \) and a positive integer \( n \) such that
   (a) \( |f(iy)| \leq M \) for all \( y \in \mathbb{R} \),
   (b) \( |f(z)| \leq C (1 + |z|^n) \) for all \( z \in \mathcal{H} \).
   Prove that \( |f(z)| \leq M \) for all \( z \in \mathcal{H} \).
   \textit{Hint:} For \( \epsilon > 0 \), consider
   \[ f_\epsilon(z) := \frac{f(z)}{(1 + \epsilon z)^{n+1}} \]
   and apply the maximum principle.