Complex Analysis Qualifying Examination

January 2004

Instructions: Please do any eight out of the nine problems listed below. You may choose to answer the problems in any order. However, to help us in grading your exam please make sure to:

i. Start each question on a new sheet of paper.
ii. Write only on one side of each sheet of paper.
iii. Number each page and write your SS# in each page.

Good luck!

1. If \( f(z) \) is analytic in the unit disc \( D = \{ z \in \mathbb{C} | |z| < 1 \} \), prove that there is an analytic function \( F(z) \) in \( D \) with \( F'(z) = f(z) \).

   You can quote the Cauchy-Goursat theorem.

2. (a) Find the Möbius transformation that sends the points \( z = 0, \infty, i \) into \( w = -1, 1, i \) respectively.

   (b) Find the image of the first quadrant under the transformation found in part (a).

3. Find the Laurent expansion of \( f(z) = (1 - z^2)^{1/2} \) around \( z = 0 \). Determine its annulus of convergence and the residue of \( f(z) \) at \( z = 0 \).

4. Compute the integral \( I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + 1)} \, dx \). Carefully justify all your steps.

5. Assume that \( f(z) \) is an entire function with \( \lim_{|z|\to\infty} \frac{f(z)}{z^2} = 0 \). Prove that \( f(z) \) must be linear, that is \( f(z) = az + b \), with \( a, b \in \mathbb{C} \). Provide all the details.

6. Assume that \( \sqrt{z} \) is given in terms of its principal branch, \( 0 \leq \arg(z) < 2\pi \). Find the image of the half-disc \( D^+ = \{(x, y) \in \mathbb{C} | x^2 + y^2 < 1, 0 < y < \infty \} \) under the map \( w = \frac{1}{2} \left( \sqrt{z} + \frac{1}{\sqrt{z}} \right) \).

7. Show that if \( f(z) \) is an entire function that never vanishes, then there is an entire function \( g(z) \) so that \( f(z) = e^{g(z)} \). Please provide all the details.

8. (a) Show that \( u(x, y) = (x + 1)y \) is harmonic in the entire plane.

   (b) Find a harmonic conjugate \( v(x, y) \) of \( u(x, y) \).

   (c) Give explicitly an analytic function \( w = f(z) \) with \( u = \text{Re} f \) and \( v = \text{Im} f \).

9. Prove the following version of the Schwarz reflection principle: if \( f(z) \) is analytic in the right plane \( \text{Re} z > 0 \), continuous in \( \text{Re} z \geq 0 \), and \( f(z) \) is purely imaginary on the imaginary axis, \( \text{Re} f(iy) = 0 \), then \( f(z) \) can be extended to an analytic function on the entire plane. Please provide all the details.