Complex Analysis Qualifying Examination

August 2002

Directions:

Do the following 8 problems. You may choose to answer the problems in any order. Please start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and write your SS # in each page. Please show all your work and explain all steps in a proof or derivation.

Questions:

1. a) Three points, \( z_1, z_2, \) and \( z_3 \) satisfy the conditions
   \[ z_1 + z_2 + z_3 = 0, \] and \[ |z_1| = 1, \quad 1 \leq |z| \leq 3. \]
   Show that these points lie at the vertices of an equilateral triangle inscribed in the unit circle.
   b) Give a generalization (without proof) of the result in part (a) for the case of \( n \) points.

2. a) Give a precise statement of the Cauchy Integral Formula.
    b) Give a proof of this formula.

3. a) State Rouche's theorem.
    b) Find the number of zeros of the function
   \[ f(z) = 2z^5 + 7z^3 + z^2 - 3 \]
   in the annulus \( 1 < |z| < 2 \)

4. Let \( f(z) \) be a complex valued continuous function on a simple contour \( \gamma \) and define a function \( g(z) \) by
   \[ g(z) = \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi \]
   a) Show that \( g(z) \) is analytic in any domain containing no points of \( \gamma \).
   b) Find an expression for \( g'(z) \).

5. a) Find a bound for the modulus of the integral shown below:
   \[ \int_{\gamma} \sin^2(z) \, dz \]
   where \( \gamma \) is the simple contour \( \gamma(t) = (1-t) + t e^{i\pi} \)
   and \( 0 \leq t \leq 1 \).
   b) Evaluate exactly the modulus of the integral in (a).
6. Classify the singularities and find residues for each of the following functions. Include points at infinity.
   a) \( z^2 + 2z^5 \)
   b) \( \frac{1}{\sin(z) - \cos(z)} \)
   c) \( \cot(z) \)
   d) \( \frac{1}{z(\exp(z) - 1)} \)

7. Evaluate the integral

\[
\int_{0}^{\pi} \frac{d\theta}{a + \sin^2(\theta)} \quad a > 0
\]

8. Evaluate the improper integral

\[
\int_{-\infty}^{\infty} \frac{\cos(x)}{1 + x^4} \, dx
\]

Include justifications for all steps in your calculation.