Reduction of Homogeneous Riemannian structures

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1 Preliminaries
   - Homogeneous Riemannian structures
   - The mechanical connection

2 Reduction
   - Reduction by a normal subgroup of isometries
   - Reduction in a principal bundle
   - Reduction of homogeneous classes and examples

3 An application: Sasakian-Kähler reduction
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A Riemannian manifold \((M, g)\) is called \textit{homogeneous} if there is a Lie group \(G\) of isometries acting transitively on it.

**Theorem (Ambrose-Singer)**

A connected, simply connected and complete Riemannian manifold \((M, g)\) is homogeneous if and only if it admits a \((1, 2)\)-tensor field \(S\) such that, if \(\nabla = \nabla - S\) where \(\nabla\) is the Levi-Civita connection, then

\[
\nabla g = 0, \quad \nabla R = 0, \quad \nabla S = 0.
\]

A tensor field \(S\) satisfying \((1)\) is called a \textit{homogeneous Riemannian structure}.
Homogeneous Riemannian structures are classified in eight invariant classes:

- **The class of symmetric spaces** \( (S = 0) \).
- **Three primitive classes**
  
  \[ S_1 = \{ S \in S \mid S_{XYZ} = g(X, Y)\varphi(Z) - g(X, Z)\varphi(Y), \varphi \in \Gamma(T^*M) \} \]
  
  \[ S_2 = \{ S \in S \mid \nabla_{XYZ} S_{XYZ} = 0, \quad c_{12}(S) = 0 \} \]
  
  \[ S_3 = \{ S \in S \mid S_{XYZ} + S_{YXZ} = 0 \}. \]

- Their direct sums \( S_1 \oplus S_2, S_1 \oplus S_3, S_2 \oplus S_3 \).
- And the generic class \( S_1 \oplus S_2 \oplus S_3 \).
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The mechanical connection

Let \((P, \bar{g})\) be a Riemannian manifold and \(\pi : P \to M\) be an \(H\)-principal bundle with \(H\) acting on \(P\) by isometries. Let \(V_{\bar{x}}P\) denote the vertical subspace at a point \(\bar{x} \in P\).

The mechanical connection is defined by the \((H\)-invariant\) horizontal distribution

\[ H_{\bar{x}}P = (V_{\bar{x}}P)_{\perp}, \quad \bar{x} \in P \]

In this situation there is a unique Riemannian metric \(g\) in \(M\) such that \(\pi_* : H_{\bar{x}}P \to T_{\pi(\bar{x})}M\) is an isometry, \(\forall \bar{x} \in P\) (we called \(g\) the reduced metric).
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Problem

\[(P, \bar{g}) \quad \bar{S} \quad (M, g) \quad ?\]
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Let \((P, \bar{g})\) be a connected homogeneous Riemannian manifold and \(\bar{G}\) a Lie group of isometries acting transitively on it. Suppose \(\bar{G}\) has a normal subgroup \(H\) acting freely on \(P\).

We consider the quotient manifold \(M = P/H\), the (left) \(H\)-principal bundle \(\pi: P \to M\) endowed with the mechanical connection, and the reduced metric \(g\) in \(M\).

In this situation the quotient group \(G = \bar{G}/H\) acts on \((M, g)\) by

\[
\pi \circ \Phi \bar{a} = \Phi a \circ \pi
\]

where \(\bar{a} \in \bar{G}, a = [\bar{a}] \in G\), and \(\Phi\) denotes the corresponding maps for the actions. This action is transitive and by isometries.
This means that \((M, g)\) is homogeneous Riemannian with \(G\) a Lie group of isometries acting transitively.

**Question**

*How homogeneous Riemannian structure tensors associated to the action of \(G\) on \(M\) are related to that of the action of \(\bar{G}\) on \(\bar{M}\)?*

\[
\begin{align*}
\bar{G} & \overset{\circ}{\longrightarrow} P & \sim & \bar{S} \\
\downarrow H & & \downarrow H & \downarrow ? \\
G = \bar{G}/H & \overset{\circ}{\longrightarrow} M = P/H & \sim & S
\end{align*}
\]
Proposition

In the previous situation, every homogeneous Riemannian structure $\tilde{S}$ associated to the action of $\tilde{G}$ induces a homogeneous Riemannian structure $S$ in $M$ associated to the action of $G$ given by

$$S_X Y = \pi_* \left( \tilde{S}_{X^H} Y^H \right) \quad X, Y \in \mathfrak{X}(M),$$

where $X^H$ denotes the horizontal lift with respect to the mechanical connection.
Proof.

Let $\tilde{g} = \tilde{m} \oplus \tilde{k}$ be a reductive decomposition of $\tilde{k}$ at $\tilde{x} \in P$ corresponding to $\tilde{S}$. From the isomorphism $\tilde{m} \to T_{\tilde{x}}P$ (given by the infinitesimal action) the mechanical connection induces an $\text{Ad}(\tilde{K})$-invariant decomposition

$$\tilde{m} = \tilde{m}^h \oplus \tilde{m}^v.$$ 

Let $\tau : \tilde{G} \to G$ be the quotient homomorphism. One proves that

$$g = \tau_\ast(\tilde{m}^h) \oplus \tilde{k}$$

is a reductive decomposition of $g$ at $\pi(\tilde{x}) \in M$.

One proves that the reduced homogeneous Riemannian structure tensor $S$ corresponding to this decomposition is

$$S_X Y = \pi_\ast \left( \tilde{S}_{X^H} Y^H \right) \quad X, Y \in \mathfrak{X}(M).$$
The set of homogeneous Riemannian structures associated to the action of $\bar{G}$ in $P$ reducing to a given homogeneous Riemannian structure $S$ associated to the action of $G$ on $M$, is in one to one correspondence with the space of $\mathfrak{ad}(\bar{K})$-equivariant maps

$$\varphi : \mathfrak{h} \to \mathfrak{\bar{k}},$$

where $\bar{K}$ is the isotropy group and $\mathfrak{\bar{k}}$ is its Lie algebra.

(One can obtain the explicit expression for those tensor fields)
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We ask under which conditions the tensor field

\[ S_X Y = \pi_* \left( \bar{S}_{X^H} Y^H \right) \quad X, Y \in \mathfrak{X}(M) \]

defines a homogeneous Riemannian structure in \((M, g)\).
Remark

- In the previous section, $H \triangleleft \tilde{G} \Rightarrow$ the mechanical connection is $\tilde{G}$-invariant $\Rightarrow$ the connection form $\omega$ is $\text{Ad}(\tilde{G})$-equivariant.

- Infinitesimally this becomes

\[
(\tilde{\nabla}_X \omega)(\tilde{Y}) = \text{ad}(\mu^{-1}(X))(\omega(\tilde{Y})) \quad X, Y \in \mathfrak{x}(P)
\]

where $\tilde{\nabla} = \tilde{\nabla} - \tilde{S}$, and $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$.

- So the covariant derivative of the connection form $\omega$ with respect to $\tilde{\nabla}$ is “proportional” to itself by a suitable linear operator.
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- Infinitesimally this becomes

$$ \left( \tilde{\nabla}_{\bar{X}} \omega \right)(\bar{Y}) = \text{ad} (\mu^{-1}(\bar{X}))(\omega(\bar{Y})) \quad \bar{X}, \bar{Y} \in \mathfrak{X}(P) $$

where $\tilde{\nabla} = \nabla - \bar{S}$, and $\tilde{\nabla}$ is the Levi-Civita connection of $\bar{g}$.

- So the covariant derivative of the connection form $\omega$ with respect to $\tilde{\nabla}$ is “proportional” to itself by a suitable linear operator.
Theorem

Let $(P, \bar{g})$ be a Riemannian manifold. Let $\pi : P \to M$ be a principal bundle with structure group $H$ acting by isometries, and endowed with mechanical connection $\omega$. For every $H$-invariant homogeneous Riemannian structure $\bar{S}$ in $P$ with $\tilde{\nabla} = \bar{\nabla} - \bar{S}$, if

$$\tilde{\nabla} \omega = \alpha \cdot \omega$$

for a certain 1-form $\alpha$ in $P$ taking values in $\text{End}(\mathfrak{h})$. Then the tensor field defined by

$$S_X Y = \pi_*(\bar{S}_{X^H} Y^H) \quad X, Y \in \mathfrak{X}(M)$$

is a homogeneous Riemannian structure in $(M, g)$. 
Proof.

- The tensor field $S$ is well-defined by $H$-invariance of $\tilde{S}$.
- For all $X, Y \in \mathfrak{X}(M)$
  \[
  \omega(\tilde{\nabla}_X Y^H) = X^H \left( \omega(Y^H) \right) - \left( \tilde{\nabla}_X \omega \right)(Y^H) \\
  = -\alpha(X^H)\omega(Y^H) = 0,
  \]
  thus $\left( \tilde{\nabla}_X Y \right)^H = \tilde{\nabla}_{X^H} Y^H$ (where $\tilde{\nabla} = \nabla - S$).
- With this one proves that $S$ satisfies Ambrose-Singer equations: equations for $g$ and $S$ are easy, but equation for the curvature $R$ is much more delicate (the curvature form of the mechanical connection appears).
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Reduction of homogeneous classes

Proposition

- The classes $S_1$, $S_3$, $S_1 \oplus S_2$ and $S_1 \oplus S_3$ are invariant under the reduction procedure.

- For the classes $S_2$ and $S_2 \oplus S_3$ (for which the trace $c_{12}$ of the homogeneous structure must vanish) one has

$$c_{12}(S)(X) = c_{12}(\bar{S})(X^{H}) - \bar{g}(H, X^{H}) \quad X \in \mathfrak{x}(M),$$

where $H$ is the mean curvature of the fibre (as a sub-Riemannian manifold of $(P, \bar{g})$) at each point.

- In particular the classes $S_2$ and $S_2 \oplus S_3$ are invariant under reduction if and only if the fibres are minimal sub-Riemannian manifolds of $(P, \bar{g})$.
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Examples

In the fibration $\mathbb{R}H(n) \to \mathbb{R}H(n - 1)$ ($\tilde{G} = \mathbb{R}H(n), H = \mathbb{R}$), the standard $S_1$ structure reduces to the standard $S_1$ structure. For $\tilde{G} = SO(n - 2)\mathbb{R}H(n), H = \mathbb{R}$, a family of structures in the generic class $S_1 \oplus S_2 \oplus S_3$ reduces to a family of structures in the generic class, and for one value of the family parameter to the standard $S_1$ structure.

In the Hopf fibrations $S^3 \to S^2$ ($\tilde{G} = U(2), H = U(1)$) and $S^7 \to \mathbb{C}P(3)$ ($\tilde{G} = U(4), H = U(1)$) (fibres are totally geodesic), a family of $S_2 \oplus S_3$ structures reduces to the symmetric case $S = 0$.

In the Hopf fibration $S^7 \to \mathbb{C}P(3)$ ($\tilde{G} = Sp(2)U(1), H = U(1)$), a 2-parameter family of $S_2 \oplus S_3$ structures reduce to a 1-parameter family of $S_2 \oplus S_3$ structures.
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Theorem (Kiričenko)

Let \((M, g)\) be a connected, simply connected and complete Riemannian manifold, and \(T_1, \ldots, T_n\) tensor fields in \(M\). Then \((M, g)\) is Riemannian homogeneous with \(T_1, \ldots, T_n\) invariant if and only if it admits a homogeneous Riemannian structure \(S\) such that \(\tilde{\nabla} T_i = 0, \ i = 1, \ldots, n.\)

- A homogeneous Riemannian structure \(S\) in an almost contact metric manifold \((M, \phi, \eta, \xi, g)\) is called an homogeneous Riemannian almost contact metric structure if \(\tilde{\nabla} \phi = 0\). If \((M, \phi, \eta, \xi, g)\) is moreover (almost) Sasakian then \(S\) is called (almost) Sasakian.

- A homogeneous Riemannian structure \(S\) in an almost Hermitian manifold \((M, J, g)\) is called an homogeneous Riemannian almost Hermitian structure if \(\tilde{\nabla} J = 0\). If \((M, J, g)\) is moreover (almost) Kähler then \(S\) is called (almost) Kähler.
Fiberings of almost contact manifolds

- **Theorem** (Ogiue): Let \((P, \phi, \xi, \eta)\) be an invariant strictly regular almost contact manifold and \(M\) the space of orbits given by \(\xi\). Then \(\pi : P \rightarrow M\) is a principal bundle, \(\eta\) is a connection form, and \(J(X) = \pi_*(\phi(X^H))\), \(X \in \mathfrak{X}(M)\) is an almost complex structure in \(M\).

- When the almost contact structure \((P, \phi, \xi, \eta, \bar{g})\) is metric the connection \(\eta\) is the mechanical connection.

- If moreover \((P, \phi, \xi, \eta, \bar{g})\) is (almost) Sasakian then \((M, J, g)\) is (almost) Kähler.

- If \(\bar{S}\) is an homogeneous almost contact metric structure then \((\bar{\nabla}\phi = 0 \Rightarrow) \bar{\nabla}\eta = 0\), so we are in the situation of the Reduction Theorem above with \(\alpha = 0, \omega = \eta\).
Proposition

If $\tilde{S}$ is a homogeneous almost contact metric structure on $(P, \phi, \xi, \eta, \tilde{g})$, then it can be reduced to a homogeneous almost Hermitian structure $S$ on $(M, J, g)$. If moreover $(P, \phi, \xi, \eta, \tilde{g})$ is (almost) Sasakian then $S$ is a homogeneous (almost) Kähler structure.
Examples:

- In the Hopf fibrations $S^3 \to S^2$ and $S^7 \to \mathbb{C}P(3)$ a family of homogeneous Sasakian structures reduces to the unique homogeneous Kähler structures $S = 0$ in $S^2$ and $\mathbb{C}P(3)$ respectively.

- A homogeneous Sasakian structure in the trivial bundle $\mathbb{C}H(n) \times \mathbb{R} \to \mathbb{C}H(n)$ reduce to a nontrivial homogeneous Kähler structure.
Future work

- Reduction the other way around

\[ (P, \bar{g}) \sim \sim \rightarrow ? \]
\[ H \]
\[ \downarrow \]

\[ (M, g) \sim \sim \rightarrow S \]

- Geometric study of the condition \( \tilde{\nabla} \omega = \alpha \cdot \omega \) (which leads to an “equivariant” version of Kiričenko’s Theorem).

- Application to symplectic/Kähler reduction (resp. hyper Kähler reduction).