4. One-Sided Variations

In certain variational problems involving an extremum of a functional $v[y(x)]$, a restriction may be imposed on the class of permissible curves that prohibits them from passing through points of

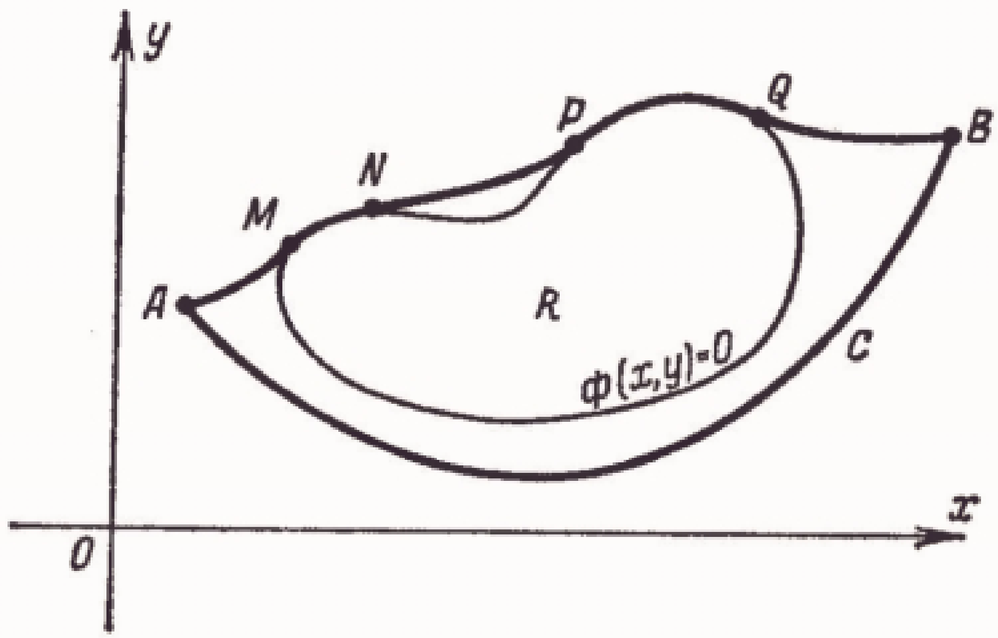

Fig. 7-14

a certain region R bounded by the curve $\Phi(x, y)=0$ (Fig 7.14). In these problems the extremizing curve C either passes completely outside the boundaries of the region R, and then it must be an extremal, since in this case the presence of the prohibited region R does not in the least affect the properties of the functional and its variations in the neighbourhood of the curve C, and the arguments in Chapter 6 hold true, or C consists of arcs lying outside the boundary of R and also consists of parts of the boundary of the region R. In this latter instance, a new situation arises; only one-sided variations of the curve C are possible on parts of the boundary of the region R, since permissible curves cannot enter the region. Parts of the curve C that lie outside the boundary of R must, as before, be extremals, since if we vary the curve C only on such a segment that permits two-sided variations, the presence of the region R will not affect the variations of y, and the conclusions of Chapter 6 continue to hold true.

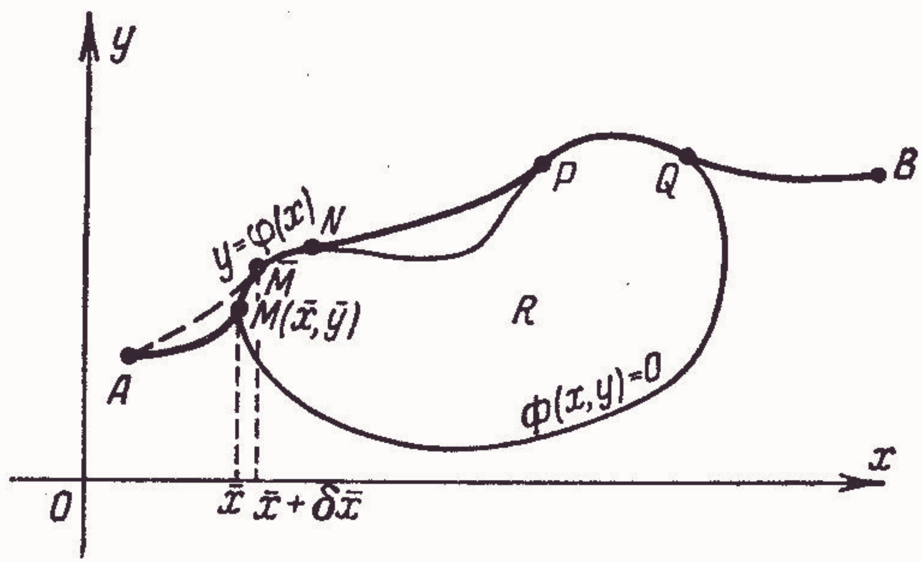

Fig. 7.15

Fig. 7-15

Thus, in the problem under consideration, an extremum can be reached only on curves consisting of arcs of extremals and parts of the boundary of the region R, and hence, in order to construct the desired extremizing curve we have to obtain conditions, at the points of transition of the extremal to the boundary of the region R, which permit determining these points. In the case depicted in Fig. 7.15, it is necessary to obtain conditions at the points M, N, P and Q. Let us, for instance, obtain a condition at the point M. In quite analogous fashion one could obtain conditions also at other points of transition of the extremal to the boundary of the region.

When calculating the variation δv of the functional

$$v=\int_{x_0}^{x_1} F(x, y, y')\,dx=\int_{x_0}^{\bar{x}} F(x, y, y')\,dx+\int_{\bar{x}}^{x_1} F(x, y, y')\,dx$$

we can consider that the variation is caused solely by the displacement of the point $M(\bar{x}, \bar{y})$ on the curve $\Phi(x, y)=0$, i.e. it may be taken that for any position of the point M on the curve

$\Phi(x, y)=0$, the arc AM is already an extremal, and the segment $MNPQB$ does not vary. The functional

$$v_1=\int_{x_0}^{\bar{x}} F(x, y, y')\,dx$$

has a boundary point moving along the boundary of the region R, whose equation is $\Phi(x, y)=0$, or in the form solved for y in the neighbourhood of the point M: $y=\varphi(x)$.

Thus, according to Sec. 1 (page 344)

$$\delta v_1=[F+(\varphi'-y')F_{y'}]_{x=\bar{x}}\,\delta\bar{x}.$$

The functional $v_2=\int_{\bar{x}}^{x_1} F(x, y, y')\,dx$ also has a moving boundary point $(\bar{x}, \bar{y})$. However, in the neighbourhood of this point the curve on which an extremum $y=\varphi(x)$ can be achieved does not vary. Consequently, the variation of the functional v_2 in the translation of point $(\bar{x}, \bar{y})$ to the position $(\bar{x}+\delta\bar{x}, \bar{y}+\delta\bar{y})$ only reduces to a change in the lower limit of integration and

$$\Delta v_2=\int^{x_1} F(x, y, y')\,dx-\int^{x_1} F(x, y, y')\,dx=$$

change in the lower limit of integration and

$$\Delta v_2 = \int_{\bar{x}+\delta\bar{x}}^{x_1} F(x, y, y')\,dx - \int_{\bar{x}}^{x_1} F(x, y, y')\,dx =$$

$$= -\int_{\bar{x}}^{\bar{x}+\delta\bar{x}} F(x, y, y')\,dx = -\int_{\bar{x}}^{\bar{x}+\delta\bar{x}} F(x, \varphi(x), \varphi'(x))\,dx,$$

since $y = \varphi(x)$ on the interval $(\bar{x}, \bar{x}+\delta\bar{x})$.

Applying the mean-value theorem and taking advantage of the continuity of the function F, we get

$$\Delta v_2 = -F(x, \varphi(x), \varphi'(x))\,|_{x=\bar{x}}\,\delta\bar{x} + \beta\cdot\delta\bar{x},$$

where $\delta \rightarrow 0$ as $\delta\bar{x} \rightarrow 0$.

Consequently, $\delta v_2 = -F(x, \varphi(x), \varphi'(x))\,|_{x=\bar{x}}\,\delta\bar{x}$,

$$\delta v = \delta v_1 + \delta v_2 = [F(x, y, y') +$$

$$+ (\varphi' - y')\,F_{y'}(x, y, y')]_{x=\bar{x}}\,\delta\bar{x} - F(x, y, \varphi')\,|_{x=\bar{x}}\,\delta\bar{x} =$$

$$= [F(x, y, y') - F(x, y, \varphi') - (y' - \varphi')\,F_{y'}(x, y, y')]_{x=\bar{x}}\,\delta\bar{x},$$

since $y(\bar{x}) = \varphi(\bar{x})$.

Due to the arbitrary nature of $\delta\bar{x}$, the necessary condition for an extremum, $\delta v = 0$, takes the form

$$[F(x, y, y') - F(x, y, \varphi') - (y' - \varphi')\,F_{y'}(x, y, y')]_{x=\bar{x}} = 0.$$

Applying the mean-value theorem, we obtain

$$(y'-\varphi')[F_{y'}(x, y, q)-F_{y'}(x, y, y')]_{x=\bar{x}}=0,$$

where q is a value intermediate between $\varphi'(\bar{x})$ and $y'(\bar{x})$. Again applying the mean-value theorem, we get

$$(y'-\varphi')(q-y')F_{y'y'}(x, y, \bar{q})|_{x=\bar{x}}=0,$$

where $\bar{q}$ is a value intermediate between q and $y'(\bar{x})$.

Suppose $F_{y'y}(x, y, \bar{q})\neq 0$. This supposition is natural for many variational problems (see Chapter 8). In this case the condition at the point M is of the form $y'(\bar{x})=\varphi'(\bar{x})$ ($q=y'$ only when $y'(\bar{x})=\varphi'(\bar{x})$, since q is a value intermediate between $y'(\bar{x})$ and $\varphi'(\bar{x})$).

Hence, at the point M the extremal AM and the boundary curve MN have a common tangent (the left tangent for the curve $y=y(x)$ and the right tangent for the curve $y=\varphi(x)$). Thus, *the extremal is tangent to the boundary of the region R at the point M.*