$\boxed{6.2.31}$ $\cancel{x^TAx}$ (27)

(old: 6.3.1)

$$A \to L_1 A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{pmatrix} \longrightarrow L_2 L_1 A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \qquad L_1 = \begin{pmatrix} 1 & & \\ 1/2 & 1 & \\ 1/2 & 0 & 1 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1 & 1 \end{pmatrix}$$

$$L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & & \\ -1/2 & 1 & \\ -1/2 & -1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & & \\ -1/2 & 1 & \\ -1/2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & & \\ & 3/2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ & 1 & -1 \\ & & 1 \end{pmatrix}$$

$$x^T A x = x^T L D L^T x = y^T D y$$

$$y = L^T x \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ x_2 - x_3 \\ x_3 \end{pmatrix}$$

$$\Rightarrow x^T A x = 2\left(x_1 - \frac{x_2}{2} - \frac{x_3}{2}\right)^2 + \frac{3}{2}\left(x_2 - x_3\right)^2$$

Set 12

6.2 (31, 33, 35, 41)

6.3 (2, 5, 8, 15, 18, 23)

$$B \to L_1 B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = U$$

$$L_1 = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ -1 & 0 & 1 \end{pmatrix} \Rightarrow B = L_1^{-1} U = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $x^T B x = (x^T L)\, D\, (L^T x)$

$$= (x_1 + x_2 + x_3)^2$$

old: 6.3.3 (new: 6.2.33) $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$: $\det(A-\lambda I)=0 \Rightarrow \lambda^2-2\lambda=0 \Rightarrow \lambda=0,2.$

To apply construction in the proof of the law of inertia,
perturb by ε: $\det(A-\varepsilon I)=0 \Rightarrow \boxed{\lambda=-\varepsilon,\ 2-\varepsilon}$

$A(\varepsilon)=\begin{pmatrix}1-\varepsilon&1\\1&1-\varepsilon\end{pmatrix}$ (If we understand $A(\varepsilon)$, we also understand A)

Now we have $A(\varepsilon)=\begin{pmatrix}1-\varepsilon&1\\1&1-\varepsilon\end{pmatrix}$

$A(\varepsilon)\,X=\lambda X \to A(\varepsilon)\begin{pmatrix}1&1\\-1&1\end{pmatrix}=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}-\varepsilon&\\&2-\varepsilon\end{pmatrix}$

$$\Rightarrow A(\varepsilon)=\begin{pmatrix}1/\sqrt2&1/\sqrt2\\-1/\sqrt2&1/\sqrt2\end{pmatrix}\begin{pmatrix}-\varepsilon&\\&2-\varepsilon\end{pmatrix}\begin{pmatrix}1/\sqrt2&-1/\sqrt2\\1/\sqrt2&1/\sqrt2\end{pmatrix}$$

$$C=\begin{pmatrix}2&0\\0&-1\end{pmatrix}=\underset{Q}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\underset{R}{\begin{pmatrix}2&0\\0&1\end{pmatrix}}$$

Then $C(t)=tQ+(1-t)QR$

$$=t\begin{pmatrix}1&0\\0&-1\end{pmatrix}+(1-t)\begin{pmatrix}2&0\\0&-1\end{pmatrix}$$

Thus $C^TAC=\begin{pmatrix}2&0\\0&-1\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}2&0\\0&-1\end{pmatrix}=\begin{pmatrix}2&2\\-1&-1\end{pmatrix}\begin{pmatrix}2&0\\0&-1\end{pmatrix}=\begin{pmatrix}4&-2\\-2&1\end{pmatrix}=A'$

$\det(A'-\lambda I)=\begin{vmatrix}4-\lambda&-2\\-2&1-\lambda\end{vmatrix}=0 \to \lambda^2-5\lambda=0$, $\lambda=0,5$
same signs

So $C(t)=\begin{pmatrix}2-t&0\\0&-1\end{pmatrix}$; cannot connect $C(t)$ to I

by nonsingular chain; some evalues must cross real axis,
$\det C<0$, $\det I=1>0$; must cross $\det C(t)=0$ to join

old: $\boxed{6.3.5}$
new: 6.2.35

$$A - \tfrac{1}{2}I = \begin{pmatrix} 2.5 & 3 & 0 \\ 3 & 9.5 & 7 \\ 0 & 7 & 7.5 \end{pmatrix}$$

$$d_1 = 2.5, \quad d_2 = \frac{\det A_2}{\det A_1} = 5.9, \quad d_3 = \frac{\det A_3}{\det A_2} = -.805..$$

$\det A_1 = 2.5 \qquad \det A_2 = 14.75 \qquad \det A_3 = -11.875$

$\Rightarrow A - \frac{1}{2}I$ has one ~~evals~~ negative and two positive eigenvalues

$\Rightarrow A$ has one evalue $< 1/2$.

$\underline{6.3.6}$ $\quad A \to x_1, \dots, x_p, \ (\lambda_i > 0) \qquad$ (x, y o.n.)

$C^TAC \to y_1, \dots, y_q \ (\mu_j \cancel{>} 0) \ \to Cy_1, \dots, Cy_q$

→ These vectors are independent: assume not, then

$$a_1 x_1 + \dots + a_p x_p = b_1 Cy_1 + \dots + b_q Cy_k = z$$

(a) $z^TAz = \sum_{i=1}^{p} \lambda_i a_i^2 \geq 0$ $\quad$ → positive

$z^TAz = (b_1 y_1^T C^T + \dots) A (b_1 Cy_1 + \dots)$

$= (b_1 y_1^T + \dots) C^TAC (b_1 y_1 + \dots) = \sum_{j=1}^{q} b_j^2 \mu_j \leq 0$ $\quad$ negative

(b) so both are zero $\Rightarrow a_i, b_j = 0$ (since $\lambda_i, \mu_j \neq 0$).

(c) $\Rightarrow$ vectors are independent. $\Rightarrow p + q \leq n$

(-) Inverting roles, find that $p' + q' \leq n$

(p': no. of Cy for ~~neg~~ positive μ, q': # of x for neg. λ)

But $p + q' = n$, $p' + q = n \Rightarrow p + q = p' + q' = n \Rightarrow p = p', \ q = q'$ $\}$ 3/12

$\boxed{6.3.11}$

new: 6.4.41

$$\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} x = \frac{\lambda}{18} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} x$$

Symmetric factorization:

$$M = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \longrightarrow \underbrace{L_1}_{= \begin{pmatrix} 1 & \\ -1/4 & 1 \end{pmatrix}} M = \begin{pmatrix} 4 & 1 \\ 0 & 15/4 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \\ 1/4 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 4 & \\ & 15/4 \end{pmatrix}}_{= \begin{pmatrix} 2 & \\ & \sqrt{15}/2 \end{pmatrix}^2} \begin{pmatrix} 1 & 1/4 \\ & 1 \end{pmatrix}$$

$$\Rightarrow M = R^T R, \quad R^T = \begin{pmatrix} 2 & 0 \\ 1/2 & \sqrt{15}/2 \end{pmatrix}$$

$$\left(R^T R = \begin{pmatrix} 2 & 0 \\ 1/2 & \sqrt{15}/2 \end{pmatrix} \begin{pmatrix} 2 & 1/2 \\ & \sqrt{15}/2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \right)$$

$$R^{T-1} = \begin{pmatrix} \sqrt{15}/2 & 0 \\ -1/2 & 2 \end{pmatrix} = C^T, \quad R^{-1} = C$$

$$\Rightarrow A x = \frac{\lambda}{18} R^T R x \quad ; \text{ let } Rx = y$$

$$\Rightarrow A R^{-1} y = \frac{\lambda}{18} R^T y \Rightarrow (C^T A C) y = \frac{\lambda}{18} y$$

$$C^T A C = \begin{pmatrix} \sqrt{15}/2 & 0 \\ -1/2 & 2 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} \sqrt{15}/2 & -1/2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3\sqrt{15} & -\frac{3}{2}\sqrt{15} \\ -9 & 27/2 \end{pmatrix} \begin{pmatrix} \sqrt{15}/2 & -1/2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 45/2 & -9\sqrt{15}/2 \\ -9\sqrt{15}/2 & 63/2 \end{pmatrix}$$

$$= \frac{9}{2} \begin{pmatrix} 5 & -\sqrt{15} \\ -\sqrt{15} & 7 \end{pmatrix}$$

$$Y_1 \sqrt{\frac{3-\sqrt{3}}{2}} + Y_2 \sqrt{\frac{3+\sqrt{3}}{2}} = 0$$

$$Y_1 \left(\frac{1+\sqrt{3}}{2}\right)\sqrt{\frac{3-\sqrt{3}}{2}} + Y_2 \left(\frac{1-\sqrt{3}}{2}\right)\sqrt{\frac{3+\sqrt{3}}{2}} = 0 \qquad \frac{Y_1 = Y_2 = 0}{\phi_1 = \phi_2 = 0}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-1+\sqrt{3}}{2} \begin{pmatrix} 1 \\ \frac{1+\sqrt{3}}{2} \end{pmatrix} \cos\omega_1 t + \frac{-1-\sqrt{3}}{2} \begin{pmatrix} 1 \\ \frac{1-\sqrt{3}}{2} \end{pmatrix} \cos\omega_2 t$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \cos\sqrt{\frac{3-\sqrt{3}}{2}}\, t - \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \cos\sqrt{\frac{3+\sqrt{3}}{2}}\, t$$

Maximum displacement for x_1:

$$\cos\omega_1 t = 1, \quad \cos\omega_2 t = -1$$

$$\omega_1 t = 2n\pi, \quad \omega_2 t = (2k+1)\pi$$

This can never happen!

$$t = \frac{(2k+1)\pi}{\omega_2} \overset{?}{=} \frac{2n\pi}{\omega_1} \Rightarrow \boxed{\frac{2k+1}{2n} \overset{?}{=} \frac{\omega_2}{\omega_1} = \sqrt{\frac{3+\sqrt{3}}{3-\sqrt{3}}}}$$

The rhs. is not rational; so this can never occur simultaneously. But the irrational number $\frac{\omega_2}{\omega_1}$ can be approximated arbitrarily closely by rationals, so the combination is achievable "in the limit". Then

$$\boxed{X_{1,max} = \left(\frac{-1+\sqrt{3}}{2}\right) + \left(\frac{1+\sqrt{3}}{2}\right) = \sqrt{3}}$$

$$\left(X_{2,max} = 1\right).$$

$$\Rightarrow \frac{9}{2}\begin{pmatrix} 5 & -\sqrt{15} \\ -\sqrt{15} & 7 \end{pmatrix} \cancel{x}y = \frac{\lambda}{18} y$$

$$\Rightarrow \underbrace{\begin{pmatrix} 5 & -\sqrt{15} \\ -\sqrt{15} & 7 \end{pmatrix}}_{\overset{\shortparallel}{B}} y = \left(\frac{\lambda}{81}\right) y$$

$$\det B = 35 - 15 = 20$$
$$\operatorname{tr} B = 12$$

$$\det(B - \mu I) = 0 \Rightarrow \mu^2 - 12\mu + 20 = 0 \Rightarrow (\mu - 2)(\mu - 10) = 0$$

$$\Rightarrow \boxed{\lambda = 810,\ 162}$$

$$(5-\mu) y_1 - \sqrt{15}\, y_2 = 0 : \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{15} \\ 5-\mu \end{pmatrix}$$

vectors: $\lambda_1 = 162,\ \vec{y}_2 = \begin{pmatrix} \sqrt{15} \\ 3 \end{pmatrix} =$

$\lambda_2 = 810,\ \vec{y}_{10} = \begin{pmatrix} \sqrt{15} \\ -5 \end{pmatrix}$

Then $y = Rx \Rightarrow x = Cy = \begin{pmatrix} \sqrt{15}/2 & -1/2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

So $\lambda_1 = 162,\ \vec{x}_1 = \begin{pmatrix} \sqrt{15}/2 & -1/2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \sqrt{15} \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

$\lambda_2 = 810,\ x_2 = \begin{pmatrix} \sqrt{15}/2 & -1/2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \sqrt{15} \\ -5 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 5 & 8 \end{pmatrix}$

Note that $x_1^T x_2 = 0$ but also $x_1^T M x_2 = 0$ so no problem.

6.3.(1/2) verify SVD & find on. bases for fundamental spaces for

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} : ① \quad AA^T = \begin{pmatrix} 17 & 34 \\ 34 & 68 \end{pmatrix} = 17 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 17 B_1$$

$$|B_1 - \lambda I| = 0 \Rightarrow (1-\lambda)(4-\lambda) - 2^2 = 0 \Rightarrow (\lambda^2 - 5\lambda) = 0 \Rightarrow \lambda = \begin{cases} 0 \\ 5 \end{cases}$$

$$(B_1 - \lambda I)\vec{u}' = 0 \Rightarrow \begin{cases} (1-\lambda) u_1' + 2 u_2' = 0 \\ 2 u_1' + (4-\lambda) u_2' = 0 \end{cases} ; \quad u_2' = 1 \Rightarrow \quad u_1' = \frac{2 u_2'}{\lambda - 1} \begin{cases} -2, & \lambda = 0 \\ 1/2, & \lambda = 5 \end{cases}$$

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} ; \quad \vec{u}_1 = \frac{2}{\sqrt{5}} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} ; \quad \sigma_1^2 = 5 \cdot 17 \; (= 17\lambda) = 85$$

② $$A^T A = \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix} = 5 \begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix} = 5 B_2$$

$$|B_2 - \lambda I| = 0 \Rightarrow (1-\lambda)(16-\lambda) - 4^2 = 0 \Rightarrow \lambda^2 - 17\lambda = 0 \Rightarrow \lambda = \begin{cases} 0 \\ 17 \end{cases}$$

$$(B_2 - \lambda I)\vec{v}' = 0 \Rightarrow \begin{cases} (1-\lambda) v_1' + 4 v_2' = 0 \\ 4 v_1' + (16-\lambda) v_2' = 0 \end{cases} \Rightarrow \quad v_2' = 1 \Rightarrow \quad v_1' = \frac{4 v_2'}{\lambda - 1} \begin{cases} v_1' = -4, & \lambda = 0 \\ v_1' = 1/4, & \lambda = 17 \end{cases}$$

$$\vec{v}_2 = \frac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \end{pmatrix} ; \quad v_1' = \frac{4}{\sqrt{17}} \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{17}} \begin{pmatrix} 1 \\ 4 \end{pmatrix} ; \quad \sigma_1^2 = 5\lambda = 5 \cdot 17 = 85$$

In both cases, the eigenvectors are orthonormal as expected.

$$\text{So}: \quad A = (\vec{u}_1, \vec{u}_2) \begin{pmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{pmatrix} (\vec{v}_1, \vec{v}_2)^T =$$

$$= \begin{pmatrix} +1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} +1/\sqrt{17} & -4/\sqrt{17} \\ 4/\sqrt{17} & 1/\sqrt{17} \end{pmatrix} = \begin{pmatrix} +1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} (\sqrt{85}) \begin{pmatrix} +1/\sqrt{17} & 4/\sqrt{17} \end{pmatrix}$$

Bases: $C(A): \left\{ \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right\} = \{u_1\}$; $C(A^T): \left\{ \begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix} \right\} = \{\vec{v}_1\}$

$\perp$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\perp$

$N(A^T): \left\{ \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \right\} = \{u_2\}$, $\quad N(A): \left\{ \begin{pmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{pmatrix} \right\} = \{\vec{v}_2\}$

6.3.5 Compute SVD for $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = B_1 ; \quad |B_1 - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) = (1-\lambda)\left[\lambda^2 - 3\lambda\right] = \lambda(\lambda-1)(\lambda-3)$$

$$(B_1 - \lambda I)\vec{v}_1 = 0 \Rightarrow \begin{cases} (1-\lambda)v_1 + v_2 = 0 \\ v_1 + (2-\lambda)v_2 + v_3 = 0 \\ v_2 + (1-\lambda)v_3 = 0 \end{cases} \Rightarrow \quad v_1 = 1: \quad v_2 = (\lambda-1)v_1, \quad v_3 = \left[(\lambda-1)(\lambda-2) - 1\right] v_1$$

$$\vec{v}_1' = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \ \lambda = 3; \quad \vec{v}_2' = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \lambda = 1; \quad \vec{v}_3' = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ \lambda = 0; \quad V = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = B_2 ; \quad |B_2 - \lambda I| = 0 = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \cancel{\lambda(\lambda+2)}$$

$$\Rightarrow \lambda = 1, 3$$

$$(B_2 - \lambda I)u = 0 \Rightarrow \left.\begin{aligned} (2-\lambda)u_1 + u_2 &= 0 \\ u_1 + (2-\lambda)u_2 &= 0 \end{aligned}\right\} \quad u_1 = 1, \quad u_2 = (\lambda-2)u_1$$

$$\vec{u}_1' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \lambda = 3; \quad u_2' = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \lambda = 1; \quad U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}}_{V^H}$$

---

note: we can remove the last Column of Σ and last Row of V^H without changing A. They are redundant - but by including them we have explicit expressions for the orthonormal bases of all the fundamental spaces

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$\begin{cases} AA^+ = I_2 \ \text{(i.e. right inverse)} \\ A^+A : \text{ortho projector onto } \operatorname{span}\{v_1, v_2\} \end{cases}$$

$$A^+ = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

6.3.8 Find $U\Sigma V^T$ if A has orthogonal columns of lengths σ_i.

$$A = (a_1\, a_2 \cdots a_n) = \underbrace{(u_1 \cdots u_n)}_{=U}\underbrace{\begin{pmatrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_n\end{pmatrix}}_{=\Sigma} \text{ with } \vec{a}_i = \sigma_i \vec{u}_i.$$

$= U\Sigma V^T$ if we take $V = I_n$, the $n\times n$ identity.

6.3.15 Find the SVD and the pseudoinverse of $V\Sigma^+ U^H$ of

(i) $A = [1\ 1\ 1\ 1]$; $A^TA = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$; $AA^T = (4)$

evalues: $\{4,0,0,0\}$ evalues $\{4\}$

nonzero eigenvalues are the same

e.vectors: A^TA $\begin{pmatrix}-3&1&1&1\\1&-3&1&1\\1&1&-3&1\\1&1&1&-3\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = \vec{0}$: $\vec{v}_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $\lambda = 4 \Rightarrow \sigma_1 = 2$

$\lambda = 0$: $(1\ 1\ 1\ 1)\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = 0$: $\vec{v}_2 = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$, $\vec{v}_3 = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$, $\vec{v}_4 = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$ (by inspection)

AA^T; $\vec{u}_1 = (1)$

$$A = (1)\,(2\ \boxed{\cancel{0}\ \cancel{0}\ \cancel{0}})\begin{pmatrix}1/2 & 1/2 & 1/2 & 1/2\\ \hline \cancel{1/2} & \cancel{1/2} & \cancel{-1/2} & \cancel{-1/2}\\ \cancel{1/2} & \cancel{-1/2} & \cancel{1/2} & \cancel{-1/2}\\ \cancel{1/2} & \cancel{-1/2} & \cancel{-1/2} & \cancel{1/2}\end{pmatrix}$$

redundant

$$A^+ = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}(1/2)(1) = \begin{pmatrix}1/4\\1/4\\1/4\\1/4\end{pmatrix}$$

$AA^+ = I_1 = (1)$ while A^+A is ortho. projector onto $C(A^T)$.

$\overline{\underline{6.3.15}}$ (ii) $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$B^TB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $\qquad BB^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

i.e. $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

obviously $B^T = B^+$ since it has the two properties of B^+:

$BB^+ = I_2$, $B^+B =$ ortho. projector $C(B^T)$

(in general $BB^+ =$ ortho. projector onto $C(B)$, here B has full column rank).

$\overline{\underline{6.3.15}}$ (iii) $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

$C^TC = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\qquad CC^T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$\lambda = \{2, 0\}$ $\qquad \lambda = \{2, 0\}$

evecs: $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$(-1)v_1 + (1)v_2 = 0 \Rightarrow v_1 = v_2 : \lambda = 2$

$v_1 + v_2 = 0 \Rightarrow v_1 = -v_2 : \lambda = 0$

$\vec{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\sqrt{2}) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\sqrt{2}) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ $\qquad C^+ = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} (1/\sqrt{2}) \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}$

$CC^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad C^+C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

ortho. projector onto $C(C)$ $\qquad$ ortho. projector onto $C(C^T)$

6.3.18 What is the minimum length least squares solution $x^+ = A^+ b$ to the following:

$$Ax = \begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}\begin{pmatrix}C\\D\\E\end{pmatrix} = \begin{pmatrix}0\\2\\2\end{pmatrix}$$

(1) By SVD:

$$AA^T = \begin{pmatrix}1&1&1\\1&1&1\\1&1&3\end{pmatrix} \qquad A^TA = \begin{pmatrix}3&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

$$|AA^T - \lambda I| = \begin{vmatrix}1-\lambda&1&1\\1&1-\lambda&1\\1&1&3-\lambda\end{vmatrix} = 0 \Rightarrow (1-\lambda)^2(3-\lambda)\underbrace{-(1-\lambda)-(3-\lambda)+1+1-(1-\lambda)}_{=3(\lambda-1)} = 0$$

Eigenvalues

$$\Rightarrow (\lambda-1)\{(\lambda-1)(3-\lambda)+3\} = (\lambda-1)\lambda(\lambda-4) = 0 \Rightarrow \lambda = \{4,1,0\}$$

$$(\sigma = +\sqrt{\lambda}) : \sigma = \{2,1,0\}$$

Eigenvectors

$$\left.\begin{aligned}(1-\lambda)u'_1 + u'_2 + u'_3 &= 0\\ u'_1 + (1-\lambda)u'_2 + u'_3 &= 0\\ u'_1 + u'_2 + (3-\lambda)u'_3 &= 0\end{aligned}\right\} u'_1 = 1$$

$u'_1 = 1$: ~~...~~ etc. ... find:

$$u'_2 = \frac{\begin{vmatrix}-1&1\\-1&3-\lambda\end{vmatrix}}{\underset{\det}{\begin{vmatrix}1-\lambda&1\\1&3-\lambda\end{vmatrix}}}, \quad u'_3 = \frac{\begin{vmatrix}1-\lambda&-1\\1&-1\end{vmatrix}}{\begin{vmatrix}1-\lambda&1\\1&3-\lambda\end{vmatrix}}$$

$$\left.\begin{aligned}(3-\lambda)v'_1 + v'_2 + v'_3 &= 0\\ v'_1 + (1-\lambda)v'_2 + v'_3 &= 0\\ v'_1 + v'_2 + (1-\lambda)v'_3 &= 0\end{aligned}\right\} \to v'_2 = 1 \quad \begin{aligned}(3-\lambda)v'_1 + v'_3 &= -v_2 = -1\\ v'_1 + (1-\lambda)v'_3 &= -v_2 = -1\end{aligned}$$

$$v'_2 = 1 : \quad v'_1 = \frac{\begin{vmatrix}-1&1\\-1&1-\lambda\end{vmatrix}}{\underset{\det}{\begin{vmatrix}3-\lambda&1\\1&1-\lambda\end{vmatrix}}}, \quad v'_3 = \frac{\begin{vmatrix}3-\lambda&-1\\1&-1\end{vmatrix}}{\begin{vmatrix}3-\lambda&1\\1&1-\lambda\end{vmatrix}}$$

$\lambda = 4 \Rightarrow \det = 2$
$\lambda = 1 \Rightarrow \det = -1$
$\lambda = 0 \Rightarrow \det = 2$

$$U' = \begin{pmatrix}1&1&1\\1&1&-1\\2&-1&0\end{pmatrix} \Rightarrow U = \begin{pmatrix}1/\sqrt6&1/\sqrt3&1/\sqrt2\\1/\sqrt6&1/\sqrt3&-1/\sqrt2\\2/\sqrt6&-1/\sqrt3&0\end{pmatrix}$$

$\|\cdot\|$: $\sqrt6$, $\sqrt3$, $\sqrt2$

$\lambda = 4 \Rightarrow \det = 2$
$\lambda = 1 \Rightarrow \det = -1$
$\lambda = 0 \Rightarrow \det = 2$

$$V' = \begin{pmatrix}2&-1&0\\1&1&1\\1&1&-1\end{pmatrix}, \quad V = \begin{pmatrix}2/\sqrt6&-1/\sqrt3&0\\1/\sqrt6&1/\sqrt3&1/\sqrt2\\1/\sqrt6&1/\sqrt3&-1/\sqrt2\end{pmatrix}$$

Then $A = U\Sigma V^T \Rightarrow$ ~~A~~ $\Sigma = \begin{pmatrix}2&&\\&1&\\&&0\end{pmatrix}$

$$= \begin{pmatrix}1/\sqrt6&1/\sqrt3\\1/\sqrt6&1/\sqrt3\\2/\sqrt6&-1/\sqrt3\end{pmatrix}\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}2/\sqrt6&1/\sqrt6&1/\sqrt6\\-1/\sqrt3&1/\sqrt3&1/\sqrt3\end{pmatrix};$$

$$A^+ = V\Sigma^+U^T = \begin{pmatrix}2/\sqrt6&-1/\sqrt3\\1/\sqrt6&1/\sqrt3\\1/\sqrt6&1/\sqrt3\end{pmatrix}\begin{pmatrix}1/2&0\\0&1\end{pmatrix}\begin{pmatrix}1/\sqrt6&1/\sqrt6&2/\sqrt6\\1/\sqrt3&1/\sqrt3&-1/\sqrt3\end{pmatrix}$$

$\Rightarrow$ over

the minimum length solution to the least squares problem

is $\quad X^+ = A^+ b = V\Sigma^+ U^T b$" $\begin{pmatrix}0\\2\\2\end{pmatrix}$ $\qquad \begin{pmatrix} 1/\sqrt6 & 1/\sqrt6 & 2/\sqrt6 \\ 1/\sqrt3 & 1/\sqrt3 & -1/\sqrt3 \end{pmatrix}\begin{pmatrix}0\\2\\2\end{pmatrix} = \begin{pmatrix}\sqrt6\\0\end{pmatrix}$

$$\Vert \qquad = V\Sigma^+\begin{pmatrix}\sqrt6\\0\end{pmatrix}, \qquad \begin{pmatrix}1/2 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\sqrt6\\0\end{pmatrix} = \begin{pmatrix}\sqrt{3/2}\\0\end{pmatrix}$$

$$\begin{pmatrix} 2/\sqrt6 & -1/\sqrt3 \\ 1/\sqrt6 & 1/\sqrt3 \\ 1/\sqrt6 & 1/\sqrt3 \end{pmatrix}\begin{pmatrix}\sqrt{3/2}\\0\end{pmatrix} = \begin{pmatrix}1\\1/2\\1/2\end{pmatrix} \qquad = V\begin{pmatrix}\sqrt{3/2}\\0\end{pmatrix} \Rightarrow$$

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(2) By the normal equations:

$$A^TA\bar{x} = A^Tb \Rightarrow \begin{pmatrix}3&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}\bar x_1\\\bar x_2\\\bar x_3\end{pmatrix} = \begin{pmatrix}4\\2\\2\end{pmatrix}$$

$$\begin{array}{c} -3 \\ -1 \\ \end{array}\left(\begin{array}{ccc|c}3&1&1&4\\1&1&1&2\\1&1&1&2\end{array}\right) \to \left(\begin{array}{ccc|c}0&-2&-2&-2\\0&0&0&0\\1&1&1&2\end{array}\right) \to \left(\begin{array}{ccc|c}0&1&1&1\\0&0&0&0\\1&1&1&2\end{array}\right) \to \left(\begin{array}{ccc|c}0&1&1&1\\0&0&0&0\\1&0&0&1\end{array}\right)$$

general solution: $\bar x_1 = 1, \ \bar x_2 + \bar x_3 = 1 \Rightarrow \bar x_2 = \sigma, \ \bar x_3 = 1-\sigma$

length: $\|\bar x(\sigma)\|^2 = 1 + \sigma^2 + (1-\sigma)^2 = 2 + 2\sigma^2 - 2\sigma$

minimum length $\frac{d\|\vec x\|^2}{d\sigma} = 4\sigma - 2 = 0 \Rightarrow \sigma = 1/2$

i.e. solution of min. length: $\bar X = \begin{pmatrix}1\\1/2\\1/2\end{pmatrix}$, exactly as in (i)

Now, the general solution to the normal equations,

$$\bar X(\sigma) = \begin{pmatrix}1\\0\\1\end{pmatrix} + \sigma\underbrace{\vec V_3}_{\text{null vector of } A^TA} = \begin{pmatrix}1\\0\\1\end{pmatrix} + 2\sigma\begin{pmatrix}0\\1/2\\-1/2\end{pmatrix}$$

$\boxed{\text{NOTE: } \vec u_3 \text{ is nullvector of } A^TA, \text{ i.e. of } A}$

To find $\bar X(\sigma)$ in the row space, demand that $\bar X(\sigma) \in N(A)^\perp$

i.e. $\langle \bar x(\sigma), \vec u_3\rangle = 0 \Rightarrow \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \vec u_3\right\rangle + 2\sigma\left\langle\begin{pmatrix}0\\1/2\\-1/2\end{pmatrix}, \vec u_3\right\rangle = 0$

$\Rightarrow -1/\sqrt2 + \cancel{2}\sigma\, 1/\cancel{2}\cdot(+2/\sqrt2) = 0 \Rightarrow \sigma = \frac12, \ \vec X(\tfrac12) = \begin{pmatrix}1\\1/2\\1/2\end{pmatrix}$ is

in row space of A.

6.3.23 Explain why AA^+ and A^+A are projection matrices (and therefore symmetric). What fundamental spaces do they project onto?

$$A^{m\times n} = U\Sigma V^H =$$

$$\begin{pmatrix} \underbrace{u_1 \cdots u_p}_{\text{basis of } C(A)} & \underbrace{u_{p+1}\cdots u_m}_{\text{basis of } N(A^H)} \end{pmatrix} \underbrace{\begin{pmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_p & & \\ 0 & & & 0 & \\ & & & & \ddots \; 0 \end{pmatrix}}_{p \text{ is the rank of } A} \begin{pmatrix} v_1^H \\ v_2^H \\ \vdots \\ v_p^H \\ v_{p+1}^H \\ \vdots \\ v_n^H \end{pmatrix} \begin{matrix} \} \text{ basis of } C(A^H) \\ \\ \} \text{ basis of } N(A) \end{matrix}$$

$$A = \begin{pmatrix} u_1 \cdots u_p \end{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_p \end{pmatrix} \begin{pmatrix} v_1^H \\ \vdots \\ v_p^H \end{pmatrix} = \hat{U}\Sigma\hat{V}^H \; ; \quad A^+ = \begin{pmatrix} v_1 \cdots v_p \end{pmatrix} \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_p \end{pmatrix} \begin{pmatrix} u_1^H \\ \vdots \\ u_p^H \end{pmatrix} = \hat{V}\Sigma^+\hat{U}^H$$

$$AA^+ = (\hat{U}\hat{\Sigma}\hat{V}^H)(\hat{V}\hat{\Sigma}^+\hat{U}^H) \qquad A^+A = (\hat{V}\hat{\Sigma}^+\hat{U}^H)(\hat{U}\hat{\Sigma}\hat{V}^H)$$

$$(\hat{V}^H\hat{V} = I^{p\times p}) \quad (\hat{\Sigma}^+\hat{\Sigma} = \hat{\Sigma}\hat{\Sigma}^+ = I^{p\times p}) \quad (\hat{U}^H\hat{U} = I^{p\times p})$$

$$AA^+ = \hat{U}\hat{U}^H \qquad A^+A = \hat{V}\hat{V}^H$$

orthogonal projectors onto

$C(A)$ $\qquad$ $C(A^H)$