Math 464
Set 6
Solutions

.3(3,6,11,12,15,18,19,20,24,25)

3.3.3 $A=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}$; $b=\begin{pmatrix}1\\1\\0\end{pmatrix}$.

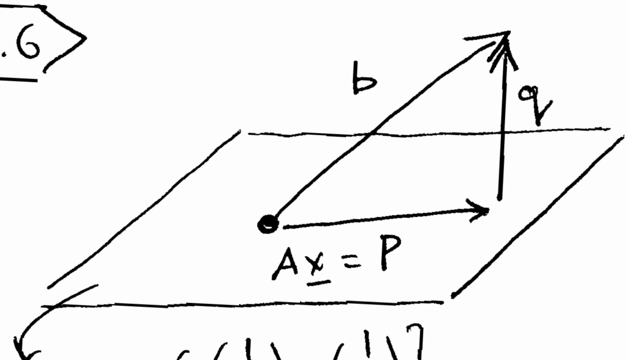
To solve $Ax=b$ by least squares, we must solve $A^TA\bar{x}=A^Tb$

$A^TA=\begin{pmatrix}2&1\\1&2\end{pmatrix}$; $A^Tb=\begin{pmatrix}1\\1\end{pmatrix}$. So $\begin{pmatrix}2&1\\1&2\end{pmatrix}\underline{x}=\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow$

$\Rightarrow\begin{pmatrix}2&1\\0&3/2\end{pmatrix}\underline{x}=\begin{pmatrix}1\\1/2\end{pmatrix}\Rightarrow\underline{x}=\begin{pmatrix}1/3\\1/3\end{pmatrix}$.

Then $p=A\underline{x}=\frac{1}{3}\begin{pmatrix}1\\1\\2\end{pmatrix}$; $b-p=\begin{pmatrix}2/3\\2/3\\-2/3\end{pmatrix}$, $\perp\ \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.

3.3.6



$b=\begin{pmatrix}1\\2\\7\end{pmatrix}$; find $\underline{x}$:

$A^TA\underline{x}=A^Tb$

$\Downarrow$

$\begin{pmatrix}9&-9\\-9&18\end{pmatrix}\underline{x}=\begin{pmatrix}-9\\27\end{pmatrix}\Rightarrow\underline{x}=\begin{pmatrix}1\\2\end{pmatrix}$.

$\mathcal{R}(A)=\text{span}\left\{\begin{pmatrix}1\\2\\-2\end{pmatrix},\begin{pmatrix}1\\-1\\4\end{pmatrix}\right\}$

$\perp$

$q=\begin{pmatrix}-2\\2\\1\end{pmatrix}$

$\Rightarrow p=A\underline{x}=\begin{pmatrix}3\\0\\6\end{pmatrix}$; $q=b-p=\begin{pmatrix}-2\\2\\1\end{pmatrix}\in\mathcal{N}(A^T)$

3.3.11 Let P project onto S; Q onto $S^\perp$. Then $(P+Q)x=Px+Qx=x$; Px projects onto S and the difference $x-Px=(I-P)x\perp Px$ so it is in $S^\perp$. By uniqueness of projection $(I-P)x=Qx\Rightarrow\boxed{P+Q=I}$

Also, PQ gives projection onto $S^\perp$ followed by projection onto S, so it must be $PQ=0$. Finally $(P-Q)(P-Q)=$
$=P^2+Q^2-PQ-QP=P+Q+0=I$. So $(P-Q)^{-1}=P-Q$.

3.3.12 $V = \text{span}\left\{ \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\}$

(a) We must find a basis for $N(A^T)$ where $A = \begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}$;

so $A^T x = 0 \Rightarrow \begin{pmatrix}1&1&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$; free variables : x_2, x_4.

(i) $x_2 = 1, x_4 = 0$: $\{A^T \vec{x} = 0 \Rightarrow x_3 = 0, x_1 = -1$: $\underline{u}_1 = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$

(ii) $x_2 = 0, x_4 = 1$: $\begin{pmatrix}1&1&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}x_1\\0\\x_3\\1\end{pmatrix} = 0 \Rightarrow x_3 = 0, x_1 = -1$; $\underline{u}_2 = \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$

and $\underline{u}_1, \underline{u}_2$ give basis of $V^\perp$.

(b) By the discussion on p. 158, the matrix $P = A(A^TA)^{-1}A^T$ projects onto $R(A)$. Here

$A = \begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}$; $A^TA = \begin{pmatrix}3&0\\0&1\end{pmatrix} \Rightarrow (A^TA)^{-1} = \begin{pmatrix}1/3&0\\0&1\end{pmatrix}$.

So $P = \begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}\begin{pmatrix}1/3&0\\0&1\end{pmatrix}\begin{pmatrix}1&1&0&1\\0&0&1&0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1&1&0&1\\1&1&0&1\\0&0&3&0\\1&1&0&1\end{pmatrix}$

(c) <u>The 0 vector</u>: An vector in $V^\perp$ projects onto $\underline{0}$ in V.

Indeed, $A^TA\,\underline{x} = A^Tb$ and $b \in V^\perp \Rightarrow b \in N(A^T) \Rightarrow A^Tb = 0$

Since A^TA is nonsingular $\Rightarrow A^TA\underline{x} = 0$ has only the solution $\underline{x} = 0$.

Then $p = A(A^TA)A^T\underline{x} \equiv 0$.

3.3.19 If $P_C = A(A^TA)^{-1}A^T$ is the projection onto the column space of A, what is the projection P_R onto the row space? (It is not P_C^T!)

(1) Recall that for P_c to be defined, we needed that A have independent columns. Then we proved A^TA is nonsingular and all worked. Essential ingredient: A has independent columns $(\underline{a}_1, \dots, \underline{a}_n) \Rightarrow \underline{\{\underline{a}_1, \dots, \underline{a}_n\} \equiv \text{basis for } \mathcal{R}(A)}$.

(2) Same will work for $A^T = (\underline{b}_1, \dots, \underline{b}_m)$ with $\underline{b}_i^T = \text{row}(A_{i*})$. If $m > n = \text{rank } A$, then pick an independent subset that forms a basis, renumber so that $B = (\underline{b}_1, \dots, \underline{b}_r)$ is a submatrix of A of size $n \times r$ (actually $r \times r$ since A had independent columns $\Rightarrow n = r$!).

Then $B(B^TB)^{-1}B^T = P_R$

(Alternatively Let $A = LU = \bar{L}\bar{U}$ (reduced factorization)

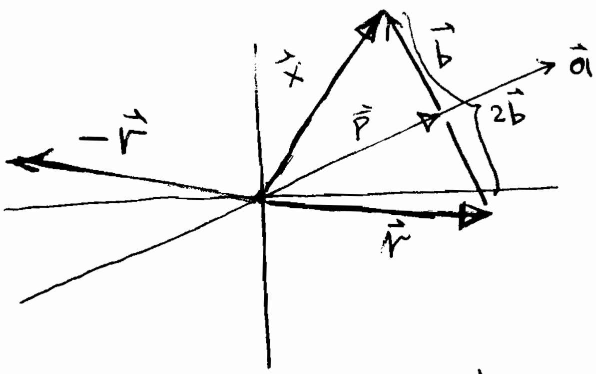
where $U = \begin{pmatrix} \bar{U} \\ 0 \end{pmatrix}$ (i.e. $U = \begin{pmatrix} \boxed{\text{staircase}} \\ 0 \end{pmatrix}$ $= \bar{U}$)

Then $\mathcal{R}(A^T) = \mathcal{R}(\bar{U})$ and

$$P_R = \bar{U}(\bar{U}^T\bar{U})^{-1}\bar{U}^T$$

3.3.15

3.3.15 If P is the projection matrix onto a line in the x-y plane, draw a figure to describe the effect of the "reflection matrix" $H = I - 2P$. Explain both geometrically and algebraically why $H^2 = I$.



$$P\vec{x} = \vec{p} \; ; \quad \vec{b} = \vec{x} - \vec{p} = (I-P)x$$

$$2\vec{b} = \vec{x} - \vec{r}$$

$$\Rightarrow \vec{r} = x - 2\vec{b}$$
$$= x - 2(x - \vec{p}) = -\vec{x} + 2\vec{p}$$

i.e. $\vec{r} = -(I - 2P)\vec{x}$

is the reflection of $\vec{x}$ by the line.

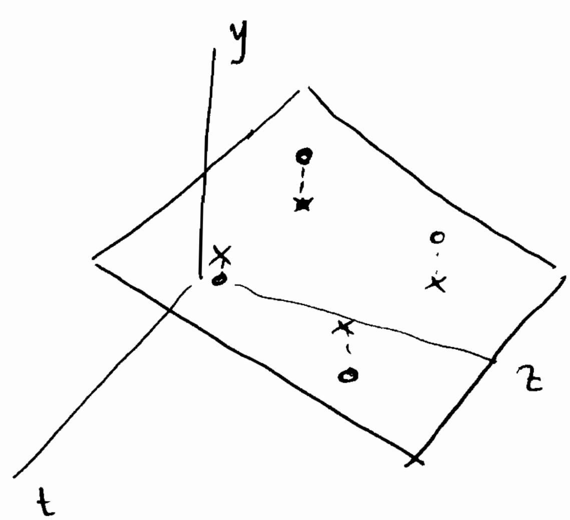
Clearly $-\vec{r} = (I - 2P)\vec{x}$ is the negative of that reflection (see figure)

3.3.18 We want to fit a plane $y = C + Dt + Ez$ to the four points

$y = 3$	at	$t = 1, z = 1$	$y = 6$	at	$t = 0, z = 3$
$y = 5$	at	$t = 2, z = 1$	$y = 0$	at	$t = 0, z = 0.$

(1) Find 4 equations in 3 unknowns to pass a plane through the points (if there is such a plane).
(2) Find 3 equations in 3 unknowns for the best least squares solution.

1) $$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 6 \\ 0 \end{pmatrix} \; ; \; Ax = b$$



2) $$A^T A \underline{x} = A^T b \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 11 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 26 \end{pmatrix}$$

3.3.20 If P_R is the projection onto the row space of A, what is the projection P_N onto the nullspace? (The two subspaces are orthogonal.)

$P_N = I - P_R$; following problem (3.3.11)

3.3.24 Find the best straight line fit to the following measurements, and sketch your solution:

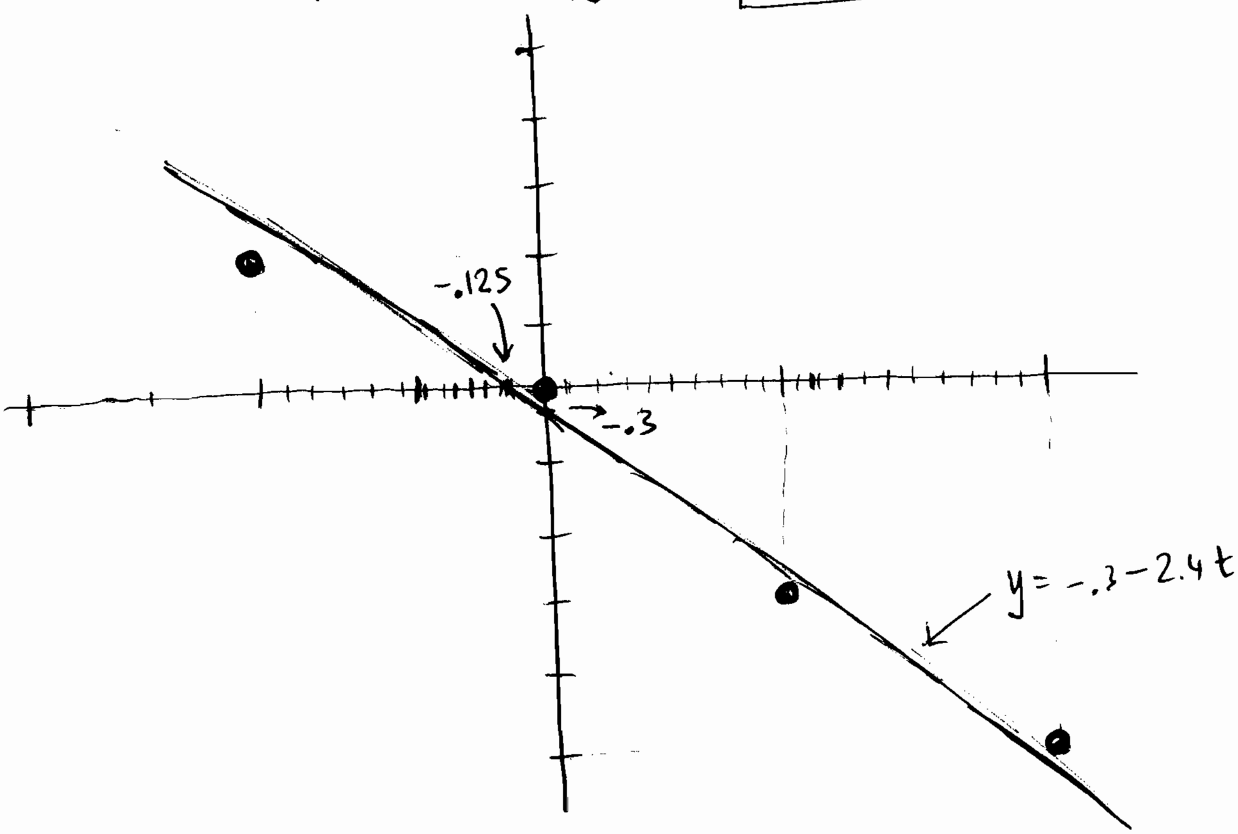
$$y = 2 \text{ at } t = -1, \qquad y = 0 \text{ at } t = 0,$$
$$y = -3 \text{ at } t = 1, \qquad y = -5 \text{ at } t = 2.$$

Need to fit $y = a + bt$;

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix} \left\{ \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \\ -5 \end{pmatrix} \right\} \Rightarrow \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -6 \\ -15 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} -6 \\ -15 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} -6 \\ -48 \end{pmatrix} = \begin{pmatrix} -.3 \\ -2.4 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

i.e. $\boxed{y = -.3 - 2.4t}$



3.3.25 Suppose that instead of a straight line, we fit the data in the previous exercise by a parabola: $y = C + Dt + Et^2$. In the inconsistent system $Ax = b$ that comes from the four measurements, what are the coefficient matrix A, the unknown vector x, and the data vector b? You need not compute $\bar{x}$.

$$\begin{array}{c} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix} \\ \begin{matrix} (1) & (t) & (t^2) \end{matrix} \end{array} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 2 \\ -3 \\ 0 \\ -5 \end{pmatrix} \\ y \end{array}$$

Math. 464
Set ~~5~~ 6
Solutions

2.185 (2,4,5,6,9,13)
3.4 (15, 18, 25)
2.205, 3.6 (8,17,20,22)

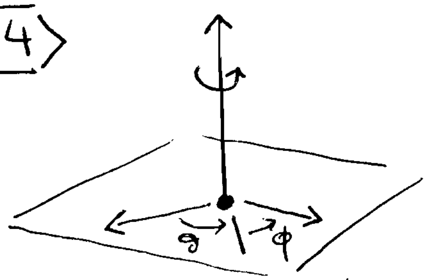
3.4.2

$P_{a_1} b = \langle a_1, b\rangle a_1$; $P_{a_2} b = \langle a_2, b\rangle a_2$

$P_{a_1,a_2} b = P_{a_1} b + P_{a_2} b$. Here $\langle a_1, b\rangle = 2$, $\langle a_2, b\rangle = 2$

So $P_{a_1,a_2} \underline{b} = 2(\underline{a}_1 + \underline{a}_2) = (1/3, 4/3, 1/3)^T$

3.4.4



For the problem to make sense, matrices must affect rotations <u>about the same axis</u>. Consider an orthogonal coordinate system in which this is the z-axis. (we assume $\mathbb{R}^3$ → in general, we could consider a rotation of the ~~xy~~ (x_1,x_2) plane in $\mathbb{R}^n$. Then

$$\Phi_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \quad \Phi_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then
$$\Phi_\theta \Phi_\phi = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi & 0 \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

On the other hand, a rotation by angle $\theta + \phi$ (combined):

$$\Phi_{\theta+\phi} = \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) & 0 \\ -\sin(\theta+\phi) & \cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equating terms, get trig. identities.

3.4.5

$$Q = I - 2uu^T \; ; \quad Q^T = I - 2uu^T = Q$$

Then $QQ^T = (I - 2uu^T)(I - 2uu^T) = I - 4uu^T + 4u\underbrace{(u^Tu)}_{=1}u^T$

$$= I \Rightarrow Q \text{ orthogonal}$$

if $\underline{u} = \left(\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\right)^T$; $Q = I - 2\begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}(1/2\ 1/2\ -1/2\ -1/2)$

$$\Rightarrow Q = I - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}(1\ 1\ -1\ -1) = I - \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} +1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix}$$

3.4.6

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ 1/\sqrt{3} & -3/\sqrt{14} \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & & \\ -1 & 1 & \\ -1 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 0 & 1/\sqrt{14} \\ 0 & -4/\sqrt{14} \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 4 & 1 \end{pmatrix}} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 0 & 1/\sqrt{14} \\ 0 & 0 \end{pmatrix}$$

so $\underbrace{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -1 & 1 & \\ -1 & 0 & 1 \end{pmatrix}}_{=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 4 & 1 \end{pmatrix}} Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 0 & 1/\sqrt{14} \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 4 & 1 \end{pmatrix} Q$: $N(Q^T) = \text{span}\left\{\begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}\right\}$

since $N(Q^T) \perp R(Q)$, any vector $\underline{n} = \alpha\begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$ is ortho. to q_1, q_2; normalizing find $q_3 = \pm\begin{pmatrix} -5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{pmatrix}$.

3.4.9

If q_1, q_2, q_3 are orthonormal, then $0q_1 + 0q_2 = 0$ is the closest vector to q_3 on span$\{q_1, q_2\}$:

Indeed $\begin{pmatrix} q_1 & q_2 \end{pmatrix} \underline{x} = q_3 \Rightarrow \begin{pmatrix} q_1^T \\ q_2^T \end{pmatrix} \begin{pmatrix} q_1 & q_2 \end{pmatrix} \underline{x} = \begin{pmatrix} q_1^T \\ q_2^T \end{pmatrix} q_3$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{\underline{x} = \underline{0}}$$

3.4.13

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$q_1 = \underline{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $q_2' = \underline{b} - \langle b, q_1 \rangle q_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = q_2$

$q_3' = c - \langle c, q_1 \rangle q_1 - \langle c, q_2 \rangle q_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - q_1 - q_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = q_3$

i.e. $q_1 = \underline{a}$; $q_2 = b - q_1$; $q_3 = c - q_1 - q_2$

or $\underline{a} = q_1$, $b = q_1 + q_2$, $c = q_1 + q_2 + q_3$

$$\Rightarrow \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{R}$$

$\boxed{3.4.15}$

To find q_3, simply find left null vector:

$$\begin{array}{c} \\ -2 \\ 2 \end{array}\begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 2 & 0 & 1 \end{pmatrix}} \begin{array}{c} \\ \\ 2 \end{array}\begin{pmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 6 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 2 & 1 \end{pmatrix}} \underset{u}{\begin{pmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \end{pmatrix}}$$

So: $\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ \boxed{-2 \;\; 2 \;\; 1} \end{pmatrix}$

i.e. $\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow N(A^T) = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right\}$

So $q_3 = \dfrac{1}{\sqrt{1+4+4}}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

To find q_1, q_2: $q_1' = \underset{\overset{\|}{a}}{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}$; $\|q_1'\| = 3 \Rightarrow q_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$$q_2' = \underset{\overset{\|}{\underline{b}}}{\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}} - \underset{\overset{\|}{\frac{-9}{3}}}{\langle q_1, \underline{b} \rangle}\, \underset{\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{q_1} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$\|q_2'\| = \sqrt{4+1+4} = 3 \quad \Rightarrow \quad q_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

So: $q_1 = \frac{1}{3}a$; $q_2' = \underline{b} + 3q_1 \Rightarrow q_2 = \frac{1}{3}q_2' = \frac{1}{3}\underline{b} + q_1$

i.e. $a = 3q_1$; $\underline{b} = -3q_1 + 3q_2 \Rightarrow (a\ \underline{b}) = (q_1\ q_2)\begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}$

$A\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}\underline{x} = \begin{pmatrix} q_1^T \\ q_2^T \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -9 \\ 18 \end{pmatrix} =$

$\Rightarrow \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}\underline{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\underline{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow \boxed{\underline{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

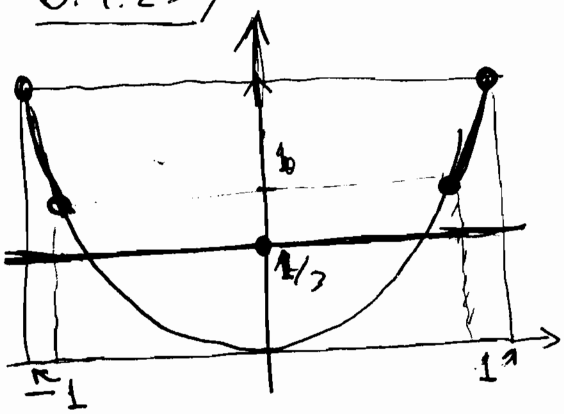
3.4.18

$$P = A(A^TA)^{-1}A^T = QR(R^TQ^TQR)^{-1}R^TQ^T$$

$$= QR(R^TR)^{-1}R^TQ^T = QR(R^{-1}(R^T)^{-1}R^T)Q^T$$

$$\boxed{P_Q = QQ^T}$$

3.4.25



$$\begin{pmatrix}1\\x\end{pmatrix}\left[\begin{pmatrix}1 & x\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = x^2\right]$$

$$\Rightarrow \begin{pmatrix}\langle 1,1\rangle & \langle 1,x\rangle\\ \langle 1,x\rangle & \langle x,x\rangle\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}\langle 1,x^2\rangle\\ \langle x,x^2\rangle\end{pmatrix}$$

$$\langle 1,1\rangle = \int_{-1}^{1} dx = 2, \quad \langle 1,x\rangle = \int_{-1}^{1} x\,dx = 0,$$

$$\langle 1,x^2\rangle = \langle x,x\rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3} \;;\; \langle x,x^2\rangle = \int_{-1}^{1} x^3 dx = 0$$

so: $$\begin{pmatrix}2 & 0\\0 & 2/3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}2/3\\0\end{pmatrix} \Rightarrow \begin{cases}a = 1/3\\ b = 0\end{cases}$$

i.e. $p(x) = 1/3$ gives the straight line closest to x^2 over the interval $(-1,1)$

3.6.8

$$V = \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right\} \qquad W = \operatorname{span}\left\{\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$$

$$V+W = \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\} := R(A) \quad \text{, where}$$

$$\Rightarrow A = \begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&1\end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix}1&1&0&0\\0&-1&1&0\\0&1&0&1\\0&0&1&1\end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix}1&1&0&0\\0&-1&1&0\\0&0&-1&1\\0&0&1&1\end{pmatrix} \rightarrow \begin{pmatrix}1&1&0&0\\0&-1&1&0\\0&0&-1&1\\0&0&0&2\end{pmatrix} \quad \text{independent vectors}$$

$\Rightarrow \dim R(A) = 4$ and the columns of A form a basis for $R(A)$.

The intersection $V \cap W$ must be the set $\{\underline{0}\}$. Since these two subspaces have mutually independent bases they cannot have a vector in common:

Assume $\underline{x} \in V \cap W$. Then

$$\left.\begin{array}{l} \underline{x} \in V \Rightarrow \underline{x} = \alpha\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \\ \underline{x} \in W \Rightarrow \underline{x} = \gamma\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \delta\begin{pmatrix}0\\0\\1\\1\end{pmatrix} \end{array}\right\}$$

but then $\alpha\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \gamma\begin{pmatrix}0\\1\\0\\1\end{pmatrix} - \delta\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$\Rightarrow \alpha = \beta = \gamma = \delta = 0$ since these vectors are an independent set.

Then $\dim(V \cap W) = 0$

3.6.17

$$A = \begin{pmatrix} 4 & 12 \\ 12 & 45 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & \\ -3 & 1 \end{pmatrix}} \begin{pmatrix} 4 & 12 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right] \left[\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 2 & 0 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$$

3.6.20

Generalizing (6): $E^2 = \sum_{i=1}^{m} w_i^2 (x - b_i)^2$

Then $\frac{dE^2}{dx} = 0 \Rightarrow 2 \sum_{i=1}^{m} w_i^2 (x - b_i) = 0 \Rightarrow \boxed{\bar{x}_w = \frac{\sum_1^m w_i^2 b_i}{\sum_1^m w_i^2}}$

3.6.22

$$Ax = b \Rightarrow WAx = Wb \Rightarrow \underbrace{(WA)^T (WA) \underline{x} = (WA)^T Wb}_{\text{normal equations}}$$

Here $WA = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\left.\begin{aligned} (WA)^T (WA) &= \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 5 \end{pmatrix} \\ Wb &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad (WA)^T Wb = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}\right\}$$

$$\begin{pmatrix} 6 & 3 \\ 3 & 5 \end{pmatrix} \underline{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\longrightarrow$

then

$$\begin{pmatrix}6&3\\3&5\end{pmatrix}\underline{x} = \begin{pmatrix}2\\3\end{pmatrix} \Rightarrow \underline{x} = \frac{1}{21}\begin{pmatrix}5&-3\\-3&6\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}$$

$$\underline{x} = \frac{1}{21}\begin{pmatrix}1\\12\end{pmatrix}$$

Now $A\underline{x} = \frac{1}{21}\begin{pmatrix}1&0\\1&1\\1&2\end{pmatrix}\begin{pmatrix}1\\12\end{pmatrix} = \frac{1}{21}\begin{pmatrix}1\\13\\25\end{pmatrix}$

and $b - A\bar{x} = \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{21}\begin{pmatrix}1\\13\\25\end{pmatrix} = \frac{1}{21}\begin{pmatrix}-1\\8\\-4\end{pmatrix}$

Then $\langle A\underline{x}_w, b-A\bar{x}_w\rangle_w \equiv [W(A\underline{x}_w)]^T[W(b-A\underline{x}_w)] =$

$$= \frac{1}{21}\begin{pmatrix}1 & 13 & 25\end{pmatrix}\begin{pmatrix}2&&\\&1&\\&&1\end{pmatrix}^2 \frac{1}{21}\begin{pmatrix}-1\\8\\-4\end{pmatrix} =$$

$$= \frac{1}{(21)^2}\cdot\begin{pmatrix}2 & 13 & 25\end{pmatrix}\begin{pmatrix}-2\\8\\-4\end{pmatrix} = \frac{1}{(21)^2}(-4+104-100) = 0$$

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p. 78, 2.2.5

$$A^T x = b \rightarrow \left(\begin{array}{cc|c} 0 & 0 & b_1 \\ 1 & 2 & b_2 \\ 4 & 8 & b_3 \\ 0 & 0 & b_4 \end{array}\right) \xrightarrow{P_{13}} \left(\begin{array}{cc|c} 4 & 8 & b_3 \\ 1 & 2 & b_2 \\ 0 & 0 & b_1 \\ 0 & 0 & b_4 \end{array}\right) \rightarrow$$

$$\xrightarrow{\begin{pmatrix} 1 & & & \\ -1/4 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}} \left(\begin{array}{cc|c} 4 & 8 & b_3 \\ 0 & 0 & b_2 - \frac{1}{4}b_3 \\ 0 & 0 & b_1 \\ 0 & 0 & b_4 \end{array}\right) \xrightarrow{\begin{pmatrix} 1/4 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}} \left(\begin{array}{cc|c} 1 & 2 & b_3/4 \\ 0 & 0 & b_2 - b_3/4 \\ 0 & 0 & b_1 \\ 0 & 0 & b_4 \end{array}\right)$$

(1) Null space $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \Rightarrow \boxed{\underline{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$ basis of $\mathcal{N}(A^T)$

(2) Row space, i.e. column space of A^T: $b_1 = 0,\ b_4 = 0,\ b_2 = \frac{1}{4} b_3 \Rightarrow b_3 = 4b_2$

i.e. $\mathcal{R}(A^T) = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 4 \\ 0 \end{pmatrix}\right\}$

y_2 free $\Rightarrow$ set $y_2 = 0$

(3) Solutions: $A^T x = b \Rightarrow U x = c \Rightarrow \begin{pmatrix} 1 & 2^* \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow y_1 = b_2,\ y_2 = 0$

i.e. $\underline{x}_{\text{particular}} = \begin{pmatrix} b_2 \\ 0 \end{pmatrix}$; $\underline{x}_{\text{homo.}} = \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\underline{x}_{\text{general}} = \begin{pmatrix} b_2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

where $\underline{b} = \begin{pmatrix} 0 \\ b_2 \\ 4b_2 \\ 0 \end{pmatrix}$, ($b_2$ arbitrary).

$\underline{3.1.12}$ Find basis for $N(A)$, $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$;

p. 142

Verify $N(A) \perp R(A^T)$; Given $x = (3,3,3)$

split as $\underline{x} = \underline{x}_r + \underline{x}_n$, $\underline{x}_r \in R(A^T)$, $\underline{x}_n \in N(A)$.

$$-1\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = -2x_3 \\ x_2 = -2x_3 \end{matrix} \rightarrow \underline{u} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

So: $N(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$; $N(A) = \text{span}\left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}$

To express a vector in terms of $\underline{x}_r \in R(A^T)$, $\underline{x}_n \in N(A)$,

we need: $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \underline{x}_r + c\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ so that $\underline{x}_r \perp \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$

Since $\underline{x}_r = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - c\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$, find c so $\underline{x}_r^T \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = 0$.

We have $\begin{pmatrix} 3+2c \\ 3+2c \\ 3-c \end{pmatrix}^T \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = (3+2c,\ 3+2c,\ 3-c)\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = 0$

$$\Rightarrow \quad -2(3+2c) - 2(3+2c) + (3-c) = 0$$

$$\Rightarrow \quad -6 - 4c - 6 - 4c + 3 - c = -9 - 9c = 0 \Rightarrow c = -1$$

i.e. $\underline{x}_r = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ $\left(\text{indeed } (1\ 1\ 4)\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = 0 \right)$

Thus
$$\boxed{\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \underset{\substack{\uparrow \\ R(A^T)}}{\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}} - \underset{\substack{\uparrow \\ N(A)}}{\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}}}$$