Solutions - Set III

Math 464/514

$$\begin{cases} 2.1\ (p.69)\ \langle 2,4,7,8\rangle \\ 2.2\ (p.77)\ \langle 4,5,7,8,14\rangle \\ 2.3\ (p.87)\ \langle 2,6,9,10,18,19\rangle \end{cases}$$

✓ **2.1.2**

Yes ≡ subspace

2.1.2 Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?
(a) The plane of vectors with first component $b_1 = 0$.
(b) The plane of vectors b with $b_1 = 1$.
(c) The vectors b with $b_1 b_2 = 0$ (this is the union of two subspaces, the plane $b_1 = 0$ and the plane $b_2 = 0$).
(d) The solitary vector $b = (0, 0, 0)$.
(e) All combinations of two given vectors $x = (1, 1, 0)$ and $y = (2, 0, 1)$.
(f) The vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

(a) Yes: $\underline{0} \in V$ and $x_1, x_2 \in V \Rightarrow y = c_1x_1 + c_2x_2 \in V$
(if x_1, x_2 has first component $=0$ so does their sum).

(b) No: ~~$x_1 \in V$~~ $\underline{0} \notin V$.

(c) If ~~$x_1 \ldots$~~ $x_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$ and $x_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}$, then
$x_1 + x_2 = \begin{pmatrix}1\\1\\\vdots\\0\end{pmatrix}$ not in V. NO

(d) $b = (0,0,0)$: $V = \{b\}$ is the trivial subspace YES

(e) $V = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$, yes.

(f) $V = \mathcal{N}(A)$ where $A = (1, \ -1 \ \ 3) \in \mathbb{R}^{1\times 3}$

The nullspace of any matrix forms a subspace of the space on which the matrix acts (i.e. $\mathbb{R}^3$ here).
YES

2.1.4

2.1.4 What is the smallest subspace of 3 by 3 matrices which contains all symmetric matrices and all lower triangular matrices? What is the largest subspace which is contained in both of those subspaces?

(a) A basis for symmetric 3x3 is given by

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

A basis for lower triangular matrices is

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}.$$

Combining these, we can get any 3x3 matrix : all $\mathbb{R}^{3\times3}$

(b) The largest subspace in both is the space of diagonal 3x3 matrices

2.1.7

2.1.7 Which of the following are subspaces of $\mathbf{R}^\infty$?

(a) All sequences like (1, 0, 1, 0, . . .) which include infinitely many zeros.
(b) All sequences $(x_1, x_2, \ldots)$ with $x_j = 0$ from some point onward.
(c) All decreasing sequences: $x_{j+1} \le x_j$ for each j.
(d) All convergent sequences: the x_j have a limit as $j \to \infty$.
(e) All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.
(f) All geometric progressions $(x_1, kx_1, k^2x_1, \ldots)$ allowing all k and x_1.

(a) $\underline{0} = (0,0,\ldots)$ is in space but ~~and~~ a linear combo of two such sequences ~~is in space~~ may not be in space :

$X_1 = (1,0,1,0,\ldots)$ (even zeroes)

$X_2 = (0,1,0,1,\ldots)$ (odd zeroes)

: $X_1 + X_2 = (1,1,1,1,\ldots)$ no zeroes.

Not a subspace since not closed.

2.1.7 (b) Yes ; $\underline{0}$ in ~~space~~ set and it is closed under addition.

NO: Let $\{x_j\}_1^\infty$ decreasing ; then $\{-|a|x_j\}_{j=1}^\infty$ is increasing for $a \neq 0$.

(c) ~~Yes~~ : ~~however the problem should read:~~

~~all non-increasing sequences $x_{j+1} \leq x_j \;\; \forall j$~~

~~Adding such sequences produces another one, and zero: ($x_j = 0$) is in space (reason we need $\leq$)~~

(d) All convergent sequences : $x_j \to x^*$ as $j \to \infty$.

Yes ($\underline{0}$ is in and adding two such sequences yields a convergent sequence)

(e) All arithmetic progressions : $x_{j+1} - x_j = x_j - x_{j-1} \;\; \forall j$

Yes ($\underline{0}$ in, closed under addition).

(f) NO : Sum of two such sequences with different k does not result in a geometric progression.

2.1.8 ✓

2.1.8 Which descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form a plane, line, point, subspace, nullspace of A, column space of A.

$A \to U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$; null vector: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$x_1 + x_2 = -1 \qquad x_1 = -2$

$-x_2 = -1 \qquad x_2 = 1$

all $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ form a line, which is $\mathcal{N}(A)$.

2.2.4

2.2.4 For the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix},$$

determine the echelon form U, the basic variables, the free variables, and the general solution to $Ax = 0$. Then apply elimination to $Ax = b$, with components b_1 and b_2 on the right side; find the conditions for $Ax = b$ to be consistent (that is, to have a solution) and find the general solution in the same form as Equation (3). What is the rank of A?

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow[L_1 = \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}]{} \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \quad \text{Echelon form}$$

Then $L_1 b = \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \end{pmatrix} = \hat{b}$

To solve $Ax = b$, b must be in $\mathcal{R}(A)$.

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$; x_2 is the basic variable, while x_1, x_3, x_4 are free.

Then the second column of A, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ generates the column space: $\mathcal{R}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$; for system to be consistent, we need $b \in \mathcal{R}(A) \Rightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow b_2/b_1 = 2 \quad (b_1 \neq 0)$.

If $b \in \mathcal{R}(A)$, i.e. $b = \begin{pmatrix} b_1 \\ 2b_1 \end{pmatrix}$, then $\hat{b} = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$

$$Ax = b \Rightarrow Ux = \hat{b} \Rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_2 = b_1 - \begin{pmatrix} 0 \\ 0 \end{pmatrix} x_1 - \begin{pmatrix} 4 \\ 0 \end{pmatrix} x_3 - \begin{pmatrix} 0 \\ 0 \end{pmatrix} x_4$$

Thus x_1, x_4 can be arbitrary, while

$$x_2 = b_1 - 4x_3$$

i.e. general solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ b_1 - 4x_3 \\ x_3 \\ x_4 \end{pmatrix}$, x_1, x_3, x_4 arbitrary.

2.2.5 ✓

2.2.5 Carry out the same steps, with b_1, b_2, b_3, b_4 on the right side, for the transposed matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix}.$$

(maximal pivot)

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \xrightarrow{P_{13}} A_1 = P_{13}A = \begin{bmatrix} 4 & 8 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[L_1]{} U = L_1 P_{13} A = \begin{bmatrix} 4 & 8 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$L_1 = \begin{pmatrix} 1 & & & \\ -1/4 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Ay = b \Rightarrow Uy = \hat{b} : \begin{pmatrix} 4 & 8 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 - 1/4\,\hat{b}_1 \\ \hat{b}_3 \\ \hat{b}_4 \end{pmatrix}$$

$\hat{b}_3 = b_1, \quad \hat{b}_1 = b_3$

Basic variables: y_1

Free variables: y_2

$\mathcal{R}(A) = \text{span}\left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$; need $\hat{b}_2 - 1/4\,\hat{b}_1 = 0, \quad \hat{b}_3 = \hat{b}_4 = 0$

Then $\begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} y_1 = \begin{pmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \end{pmatrix} y_2 \Rightarrow y_1 = \frac{b_1}{4} - 2y_2$

General solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \cancel{\begin{pmatrix} y_1 \\ \frac{b_1}{4} - 2y_2 \end{pmatrix}} \begin{pmatrix} \frac{b_3}{4} - 2y_2 \\ y_2 \end{pmatrix}$

2.2.7 ✓

2.2.7 Describe the set of attainable right sides b for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank? How many free variables, and how many solutions?

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix} \xrightarrow[L_1]{} A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} \xrightarrow[L_2]{} \cancel{A_2}\ U = L_2 L_1 A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -2 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -3 & 1 \end{pmatrix}$$

Then $Ax = b \longrightarrow Ux = \hat{b}$

$$\hat{b} = L_2 L_1 \hat{b} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 \end{pmatrix}$$

$$= \begin{pmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 - 3b_2 \end{pmatrix}$$

The rank is the same as the number of pivots, i.e.

$\text{rank}(A) = 2$.

Basic variables: u, v

Free variables: None

Constraints for solvability: $b_3 - 2b_1 - 3b_2 = 0$

If $b \in \mathcal{R}(A)$ (i.e. if constraint is satisfied) then solution is unique:

$$Ux = \hat{b} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} u = b_1 \\ v = b_2 \\ (b_3 = 2b_1 + 3b_2) \end{matrix}}$$

2.2.8 √

2.2.8 Find the value of c which makes it possible to solve

$$u + v + 2w = 2$$
$$2u + 3v - w = 5$$
$$3u + 4v + w = c.$$

$Ax = b$:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{pmatrix} \xrightarrow{L_1 = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ -3 & 0 & 1 \end{pmatrix}} A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{pmatrix} \xrightarrow{L_2 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix}} A_2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U = L_2 L_1 A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \hat{b} = L_2 L_1 b = L_2 \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 3b_1 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - b_2 - b_1 \end{pmatrix}$$

$Ax = b \rightarrow Ux = \hat{b}$

Thus $U x = \hat{b}$; here $b_1 = 2,\ b_2 = 5,\ b_3 = c$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ c-7 \end{pmatrix} \quad ; \text{ Need } \underline{c = 7}$$

Then, u, v : basic, w free

$u + v = 2 - 2w$

$v = 3 + 5w$

$\Rightarrow u = -1 - 7w$

$$\Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} w \quad \text{is solution.}$$

2.2.14

2.2.14 Write down a 2 by 2 system $Ax = b$ in which there are many solutions $x_{\text{homogeneous}}$ but no solution $x_{\text{particular}}$. Therefore the system has no solution.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad ; \quad b \notin \mathcal{R}(A), \quad \mathcal{N}(A) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

so $x_{hom} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ but no solution.

2.3.2 √

2.3.2 Decide the dependence or independence of

(a) (1, 1, 2), (1, 2, 1), (3, 1, 1);

(b) $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_1$ for any vectors v_1, v_2, v_3, v_4;

(c) (1, 1, 0), (1, 0, 0), (0, 1, 1), (x, y, z) for any numbers x, y, z.

To decide about the dependence/independence of a set of vectors we simply determine the rank of a matrix they form, as columns or as rows (same answer).

(a) (columns)

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow[L_1 = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ -2 & 0 & 1 \end{pmatrix}]{} A_1 = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{pmatrix} \xrightarrow[L_2 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1 & 1 \end{pmatrix}]{}$$

$$\rightarrow U = L_2 L_1 A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{pmatrix} \quad ; \quad 3 \text{ pivots} \Rightarrow \operatorname{rank}(A) = 3$$

$\Rightarrow$ <u>independent</u>

(b) $B = \begin{pmatrix} v_1 - v_2, & v_2 - v_3, & v_3 - v_4, & v_4 - v_1 \end{pmatrix} = \underbrace{\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{A}$

We just need to decide whether A is nonsingular, because if it is then the rank of B is the same as the rank of V (i.e. the vectors $v_1 - v_2, \dots, v_4 - v_1$ are independent just like v_1, v_2, v_3, v_4.

Now: $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

work with A^T: $A^T \xrightarrow[L_1 = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 1 & 0 & 0 & 1 \end{pmatrix}]{} A_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \xrightarrow[L_2 = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & 1 \end{pmatrix}]{}$

$\to A_2 = L_2 L_1 A^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow[L_3 = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 1 & 1 \end{pmatrix}]{}$

$\to U = L_3 L_2 L_1 A^T = \begin{pmatrix} 1 & -1 & & \\ 0 & 1 & -1 & \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; rank U = rank A^T = rank A = 3

$\Rightarrow$ $(v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_1)$ are dependent even if (v_1, v_2, v_3, v_4) are not.

(c) $\begin{pmatrix} 1 & 1 & 0 & x \\ 1 & 0 & 1 & y \\ 0 & 0 & 1 & z \end{pmatrix}$ are dependent for any x, y, z

since the rank of any 3×4 matrix is at most $r \le \min(m,n)$, here $r \le 3$. (indeed $r=3$).

2.3.6 Describe geometrically the subspace of $\mathbf{R}^3$ spanned by
(a) (0, 0, 0), (0, 1, 0), (0, 2, 0);
(b) (0, 0, 1), (0, 1, 1), (0, 2, 1);
(c) all six of these vectors. Which two form a basis?
(d) all vectors with positive components.

2.3.6 ✓

(a) $A: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$; $\mathcal{R}(A) = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ (already in echelon form)

(b) $B: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{P_{13}} B_1 = P_{13}B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$: pivots on cols. 1, 2.

$\mathcal{R}(B) = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

(c) The subsbace V spanned by all six, is spanned by the union of the bases of $\mathcal{R}(B)$, $\mathcal{R}(A)$, i.e.

$$V = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

(d) The space is spanned by $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$, i.e. $= \mathbb{R}^3$

2.3.9

2.3.9 By locating the pivots, find a basis for the column space of

$$U = \begin{bmatrix} 0 & 1 & 4 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Express each column that is not in the basis as a combination of the basic columns. Find also a matrix A with this echelon form U, but a different column space.

$$\mathcal{R}(U) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}$$ (pivots on 2^{nd}, 3^{rd} columns)

Need x, y: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x + \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} y = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad x + 4y = 3 \qquad \Rightarrow x = 3 - 4 = -1$

$\qquad 2y = 2 \Rightarrow y = 1$

i.e. $\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

To get some A with U as its echelon form, just need some (any) L:

$A = LU$; let $A = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 0 & 1 & \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 4 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 4 & 3 \\ 0 & 1 & 6 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 3 \end{pmatrix}$

$\Rightarrow \mathcal{R}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 4 \\ 4 \end{pmatrix} \right\}$

2.3.10

2.3.10 Suppose we think of each 2 by 2 matrix as a "vector." Although these are not vectors in the usual sense, we do have rules for adding matrices and multiplying by scalars, and the set of matrices is closed under these operations. Find a basis for this vector space. What subspace is spanned by the set of all echelon matrices U?

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$; basis: $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Echelon matrices $\in$ span$\{e_1, e_2, e_4\}$.

2.3.18

2.3.18 *True or false*: (a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.
(b) A 5 by 7 matrix never has linearly independent columns.

(a) This is only true if A is a square matrix.
False in general

(b) $A \in \mathbb{R}^{5\times 7}$; there are 7 columns, but they live in $\mathbb{R}^5$.
Any 7 vectors in $\mathbb{R}^5$ must be linearly dependent
True

2.3.19

2.3.19 Suppose n vectors from $\mathbf{R}^m$ go into the columns of A. If they are linearly independent, what is the rank of A? n If they span $\mathbf{R}^m$, what is the rank? m If they are a basis for $\mathbf{R}^m$, what then? $m=n$

$A \in \mathbb{R}^{m\times n}$; if columns are independent $\Rightarrow n \le m$ and $\text{rank}(A) = n$.

If ~~span{col~~ $\mathcal{R}(A) = \mathbb{R}^m \Rightarrow \text{rank}(A) = m$; $m \le n$

If columns of A are basis $\Rightarrow m = n$, $\text{rank}(A) = m = n$.

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