Math. 464
Fall 2K1
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Solutions #2

p.39, 1.5(4,9,11,13,18)
p.49, 1.6(10,11)
p.60, 1.R(4,5,12,13,18)

1.5.4, p.39 Produce LU by elimination if A is

(i) $A = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$; (ii) $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$; (iii) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{pmatrix}$

(i) $A_1 = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix} \longrightarrow A_2 = L_1 A_1 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$; $L_1 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$

$L = L_1^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$; $U = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

(ii) $A_1 = \begin{array}{c} \\ -1/3 \\ -1/3 \end{array}\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{L_1} A_2 = \begin{array}{c} \\ \\ -1/4 \end{array}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{pmatrix} \xrightarrow{L_2} A_3 = U = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{pmatrix}$

$L = L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{pmatrix}$ by inspection

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(iii) $A_1 = \begin{array}{c} \\ -1 \\ -1 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{pmatrix} \xrightarrow{L_1} A_2 = \begin{array}{c} \\ \\ -1 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{pmatrix} \xrightarrow{L_2} A_3 = U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

$L = L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

1.5.9, p.40 If $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ is nonsingular

↳ see back

(a) then it must have non-zero pivots, i.e. $d_1 * d_2 * d_3 \neq 0$.

(b) To solve $Ax = b$ we write $(LDU)x = b$ so that $Lc = b$ with $DUx = c$.

$U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $U_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then, with $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $c = L^{-1} b = (L_1^{-1} L_2^{-1})^{-1} b = L_2 L_1 b$

i.e. $c = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & +1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 0 & 1 \end{pmatrix} b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$x = U^{-1} D^{-1} c = (U_2 U_3)\begin{pmatrix} 1/d_1 & & \\ & 1/d_2 & \\ & & 1/d_2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1/d_3 \end{pmatrix} = \frac{1}{d_3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

1.5.9 a) details : we compute A^{-1} directly:

$$A = LDU \Rightarrow A^{-1} = U^{-1}D^{-1}L^{-1}$$

Here $L = \begin{pmatrix}1&0&0\\-1&1&0\\0&-1&1\end{pmatrix} = L_1^{-1}L_2^{-1} = \begin{pmatrix}1&&\\-1&1&\\0&0&1\end{pmatrix}\begin{pmatrix}1&&\\0&1&\\0&-1&1\end{pmatrix}$

$$L^{-1} = L_2L_1 = \begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\1&1&0\end{pmatrix}$$

$$D^{-1} = \begin{pmatrix}1/d_1&0&0\\0&1/d_2&0\\0&0&1/d_3\end{pmatrix} \text{ exists iff } d_1\cdot d_2\cdot d_3 \neq 0$$

$$U = \begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix} = L^T \Rightarrow U^{-1} = (L^{-1})^T = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

or $U = U_3^{-1}U_2^{-1} = \begin{pmatrix}1&0&0\\0&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix}$

$$\Rightarrow U^{-1} = U_2U_3 = \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix} = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

1.5.11, p.40 Solve $LUx = b \Rightarrow \begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}2&4&4\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix} = \begin{pmatrix}2\\0\\2\end{pmatrix}$

i.) $Lc = b \Rightarrow c = L^{-1}b = \begin{pmatrix}1&0&0\\-1&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}2\\0\\2\end{pmatrix} = \begin{pmatrix}2\\-2\\0\end{pmatrix}$

L ~~simple~~ elementary triang. factor.

ii) $Ux = c$; $U = DU_3^{-1}U_2^{-1} = \begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix} = \underset{D}{\begin{pmatrix}2&&\\&1&\\&&1\end{pmatrix}}\underset{U_3^{-1}}{\begin{pmatrix}1&0&2\\0&1&2\\&&1\end{pmatrix}}\underset{U_2^{-1}}{\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}}$

$$U^{-1}c = U_2U_3D^{-1}c = \underbrace{\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}}_{U_2}\underbrace{\begin{pmatrix}1&0&-2\\0&1&-2\\0&0&1\end{pmatrix}}_{U_3}\begin{pmatrix}1/2&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2\\-2\\0\end{pmatrix}$$

$$= U_2\,U_3\begin{pmatrix}1\\-2\\0\end{pmatrix}$$

$$= U_2\begin{pmatrix}1\\-2\\0\end{pmatrix}$$

$$\Rightarrow \boxed{x = \begin{pmatrix}5\\-2\\0\end{pmatrix}}$$

1.5.13, p.41 Solve using elimination + row exchanges.

(i) $(A_1|b) = \left(\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ -2 & -8 & 3 & 32 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 0 & 0 & 1 \end{pmatrix}]{L_1} (A_2|b_2) = \left(\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 0 & 7 & 28 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{P_{23}}$

$\to (A_2'|b_2') = \left(\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 7 & 28 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1/7 \end{pmatrix}]{D^{-1}} (A_3|b_3) = \left(\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & 0 & -2 \\ & 1 & -1 \\ & & 1 \end{pmatrix}]{U_3}$

$\to (A_4|b_4) = \left(\begin{array}{ccc|c} 1 & 4 & 0 & -10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{U_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right) \Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

Permutation: P_{23}

(ii) $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow{P_{12}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -1 & 0 & 1 \end{pmatrix}]{L_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}]{U_3}$

$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{U_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Permutation: P_{12}

1.5.18, p.41 (a b): $\left(\begin{array}{ccc|cc} 0 & 1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 1 & 0 & -1 & 2 & 0 \end{array}\right) \xrightarrow{P_{12}} \left(\begin{array}{ccc|cc} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 1 & 0 & -1 & 2 & 0 \end{array}\right) \xrightarrow{L_1} \quad L_1 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -1 & 0 & 1 \end{pmatrix}$

$\to \left(\begin{array}{ccc|cc} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right) \xrightarrow[= \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix}]{L_2} \left(\begin{array}{ccc|cc} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & \boxed{-2} & 0 \end{array}\right)$

(a) (b)

(a) Inconsistent

(b) Solvable w. infinitely many sols

(c) $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{P_{12}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{L_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{L_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right)$

(c) solvable with a unique solution.

1.6.10, p.50 > Find inverses

(i) ~~Since the first row of A is (0, 0, 0, 1), it follows that the last row of A^{-1} is (1, 0, 0, 0) (This is the only way for AA^{-1} to have first row~~

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} = P_{14} P_{23} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P_{14} P_{23} D$$

$$\Rightarrow A^{-1} = D^{-1} P_{23} P_{14} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P_{23} P_{14} = \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

same as if A was diagonal! (it is, but in "crypto" form!).

(ii)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ -1/2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix}}_{L_3}$$

$$A^{-1} = L_3^{-1} L_2^{-1} L_1^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 3/4 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 2/3 & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ 1/2 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1/2 & 1 & & \\ & 2/3 & 1 & \\ & & 3/4 & 1 \end{pmatrix}$$

(iii)

$$A = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{pmatrix}; \quad A^{-1} = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} & O \\ O & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \end{pmatrix}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

i.e. $$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b & 0 & 0 \\ -c & a & 0 & 0 \\ 0 & 0 & d & -b \\ 0 & 0 & -c & a \end{pmatrix}$$

1.6.11, p.50 > (i) A, B invertible but A + B singular: $A = -B = I$

(ii) A + B invertible; A, B singular: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = I$

(iii) A, B, A+B invertible $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

If $A^{-1}, B^{-1}, (A+B)^{-1}$ exist then $A(B^{-1} + A^{-1})B = A + B \Rightarrow$

$$\Rightarrow B^{-1} + A^{-1} = A^{-1}(A+B)B^{-1} \Rightarrow \boxed{[B^{-1} + A^{-1}]^{-1} = B(A+B)^{-1}A}$$

R.1.4, p.60

1.5

(a) $(A_1|b_1) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right) \xrightarrow[\;=\begin{pmatrix} 1 & & \\ -1 & 1 & \\ -1 & 0 & 1 \end{pmatrix}]{L_1} (A_2|b_2) \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right) \xrightarrow[\;=\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix}]{L_2}$

$\rightarrow (A_3|b_3) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow[\;=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{U_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \Rightarrow \boxed{\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix}}$

Then $A = \cancel{U} \cdot L_1^{-1} L_2^{-1} U_3^{-1} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overset{L}{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{pmatrix}} \overset{U}{\begin{pmatrix} 1 & 0 & \cdots \\ & 1 & \cdots \\ & & \end{pmatrix}}$

(b) $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 6 \end{array}\right) \xrightarrow{P_{12}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 6 \end{array}\right) \xrightarrow[\;=\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -1 & 0 & 1 \end{pmatrix}]{L_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 6 \end{array}\right) \xrightarrow[\;=\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix}]{L_2}$

$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 6 \end{array}\right) \xrightarrow[\;=\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1/2 \end{pmatrix}]{D^{-1}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow[\;=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}]{U_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right)$

$\boxed{\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}}$

Then $I = (U_3 D^{-1} L_2 L_1) P_{12} A \Rightarrow$

$P_{12} A = (L_1^{-1} L_2^{-1})(D U_3^{-1})$

$= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}^{L} \overbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}}^{U}$

R.1.12, p.60

(1) TRUE: if $B = PA \Rightarrow B^{-1} = A^{-1}P^{-1}$, P any permutatio[n]

(2) FALSE in general: $A = A^T$, $B = B^T$; But $(AB)^T = B^T A^T = BA \neq AB$

(except for commuting matrices, which are exceptional!)

(3) TRUE: $(AB)^{-1} = B^{-1}A^{-1}$

(4) FALSE: $PA = LU$ is guaranteed for some P. But this factorization may be impossible for A without permutations.

R.1.13, p.61 Solve $Ax=b$ by solving $Lc=b$ & $Ux=c$ for

$$A = LU = \begin{pmatrix}1&0&0\\4&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}2&2&4\\0&1&3\\0&0&1\end{pmatrix} = \underset{L_1^{-1}}{\begin{pmatrix}1&&\\4&1&\\1&0&1\end{pmatrix}}\underset{D}{\begin{pmatrix}2&&\\&1&\\&&1\end{pmatrix}}\underset{U_3^{-1}}{\begin{pmatrix}1&0&2\\0&1&3\\0&0&1\end{pmatrix}}\underset{U_2^{-1}}{\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}}$$

i.e. $A^{-1} = \overbrace{U_2 U_3 D^{-1}}^{U^{-1}} L_1$

(i) For $Lc=b$: $\quad L_1 c = \begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow c = L_1^{-1}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

(ii) For $Ux=c \Rightarrow \cancel{x}\ x = U^{-1}c = \begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&-2\\0&1&-3\\0&0&1\end{pmatrix}\begin{pmatrix}1/2&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\Rightarrow x = \begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-2\\-3\\1\end{pmatrix} = \begin{pmatrix}1\\-3\\1\end{pmatrix} = A^{-1}b = (\underline{a}^{-1})_3 \text{ i.e. the third column of } A^{-1}.$$

R.1.18, p.61 Here $A = I + \begin{pmatrix}v_1\\v_2-1\\v_3\\v_4\end{pmatrix}\begin{pmatrix}0&1&0&0\end{pmatrix}$ (using elementary matrices from the lectures)

This is of form $A = E(\underline{w}, \underline{e}_2; 1)$ with $w = \begin{pmatrix}v_1\\v_2-1\\v_3\\v_4\end{pmatrix}$

So its inverse is $A^{-1} = E(\underline{w}, \underline{e}_2; s)$ with $\frac{1}{s} - 1 = e_2^T w = v_2 - 1$

i.e. $s = 1/v_2$ and $A^{-1} = E(w, \underline{e}_2; 1/v_2) = I - \frac{1}{v_2}\begin{pmatrix}v_1\\v_2-1\\v_3\\v_4\end{pmatrix}\begin{pmatrix}0&1&0&0\end{pmatrix}$

Or, just factoring:

To factor:

$$\begin{pmatrix}1&v_1&0&0\\0&v_2&0&0\\0&v_3&1&0\\0&v_4&0&1\end{pmatrix} = \begin{pmatrix}1&&&\\&1/v_2&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&v_1&0&0\\0&1&0&0\\0&v_3&1&0\\0&v_4&0&1\end{pmatrix} = D\cdot\begin{pmatrix}1&&&\\0&1&&\\0&v_3&1&\\0&v_4&0&1\end{pmatrix}\begin{pmatrix}1&v_1&0&0\\0&1&0&0\\&&1&0\\&&&1\end{pmatrix}$$

$$= \overset{D}{\begin{pmatrix}1&&&\\&1/v_2&&\\&v_3/v_2&1&\\&v_4/v_2&0&1\end{pmatrix}}\begin{pmatrix}1&v_1&&\\&1&0&0\\&&1&0\\&&&1\end{pmatrix} = \underset{L}{\begin{pmatrix}1&&&\\0&1&&\\0&v_3/v_2&1&\\0&v_4/v_2&0&1\end{pmatrix}}\underset{D}{\begin{pmatrix}1&&&\\&1/v_2&&\\&&1&\\&&&1\end{pmatrix}}\underset{U}{\begin{pmatrix}1&v_1&&\\&1&0&\\&&1&\\&&&\end{pmatrix}}$$

$$\underset{U^{-1}D^{-1}L^{-1}}{A^{-1}} = \begin{pmatrix}1&-v_1&0&0\\&1&0&0\\&&1&0\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\&v_2&0&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\0&1&&\\0&-v_3/v_2&1&\\0&-v_4/v_2&0&1\end{pmatrix} = \begin{pmatrix}1&-v_1/v_2&0&0\\0&v_2&0&0\\0&-v_3/v_2&1&0\\0&-v_4/v_2&0&1\end{pmatrix}$$

$$\boxed{A^{-1} = I - \frac{1}{v_2}\underline{w}\,\underline{e}_2^T}$$