Math. 464
Fall 2K1
E.A.Coutsias
Solutions, H1

1.2(1,3,8,9)
1.3(1,3,11) 1
1.4(4,21)

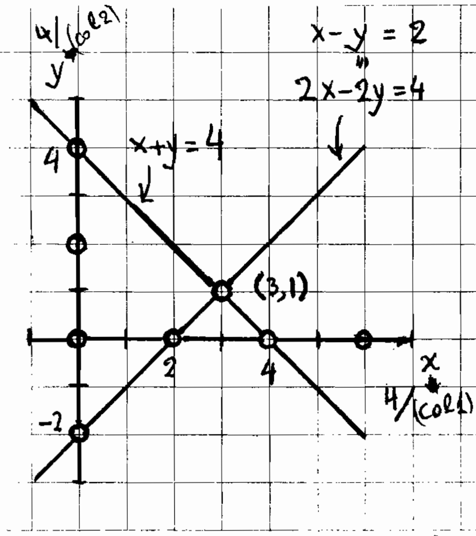
1.2.1 (p.9) Draw row & column pictures for system

$$\begin{pmatrix}1&1\\2&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}$$

Solution $\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{-1\cdot2-1\cdot2}\begin{pmatrix}-2&-1\\-2&1\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$

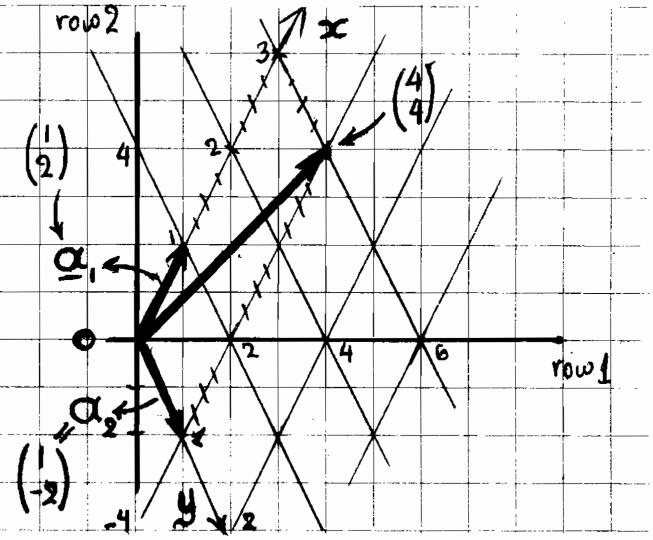
note that $\det A = 1\cdot(-2)-1\cdot 2=-4<0$ (*)

Row Picture



(1a)

Column Picture



(1b)

(*) This implies that we can view this system $Ax=\bar{b}$ as a change of coordinates with $\begin{pmatrix}x\\y\end{pmatrix}$ giving the coordinates of the point $\bar{b}=\begin{pmatrix}4\\4\end{pmatrix}$ in $\mathbb{R}^2$ in terms of a coordinate system based on the column vectors of A, $a_1=\begin{pmatrix}1\\2\end{pmatrix}$ and $a_2=\begin{pmatrix}1\\-2\end{pmatrix}$. The determinant of A, $\det A=-4$ gives: (i) The magnification of this map (notice that one unit coordinate parallelogram has area $4=|\det A|$) and (ii) The change in orientation $(-1)=\operatorname{sgn}\det A$, since the vectors a_1, a_2 define a left-handed grid

1.2.3 (p.10) Planes $\begin{cases} u+v+w+z=6 \\ u+w+z=4 \\ u+w=2 \end{cases}$

(a) Find intersection

(b) Find intersection with $u = -1$

Carry out elimination for augmented matrix

$$A = \begin{array}{c} \\ -1 \\ -1 \\ -1 \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \\ \hline 1 & 0 & 0 & 0 & -1 \end{array}\right) \to \begin{array}{c} \\ \\ -1 \\ -1 \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & -4 \\ \hline 0 & -1 & -1 & -1 & -7 \end{array}\right) \to$$

$$\downarrow$$

For (b), we find the point:

$(u,v,w,z) = (-1, +2, +3, +2)$ $\Leftarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & -2 \\ \hline 0 & 0 & -1 & 0 & -3 \end{array}\right)$

(see (a) below)

For (a) we get: $z=+2, v=+2$

and $(u+w)+(v+z) = u+w+4 = 6 \Rightarrow u+w=2.$

The three equations $(u = 2-w,\ v=+2,\ w,\ z=+2)$ define a 1-parameter subspace of $\mathbb{R}^4$, i.e. a line. In (b) we have one additional relationship ($u=-1$) fixing a point on the line.

1.2.8 (p.10) Study solvability for $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ \alpha \end{pmatrix}$

Comment on case $\alpha = 0$.

Elimination

$$\begin{array}{c} \\ -1 \\ 0 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & \alpha \end{array}\right) \xrightarrow{(1)} \begin{array}{c} \\ \\ -1 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & \alpha \end{array}\right) \xrightarrow{(2)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & \alpha+1 \end{array}\right)$$

solvable only if $\alpha = -1$; (1-parameter family of solutions).

$$\begin{pmatrix} \underline{a}^1 \\ \underline{a}^2 \\ \underline{a}^3 \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} \underline{a}^1 \\ \underline{a}^2 - \underline{a}^1 \\ \underline{a}^3 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} \underline{a}^1 \\ \underline{a}^2 - \underline{a}^1 \\ \underline{a}^3 - \underline{a}^2 + \underline{a}^1 \end{pmatrix}$$

<u>Inconsistent for $\alpha = 0$</u>

I.e. $\underline{a}^3 = \underline{a}^2 - \underline{a}^1$ and the matrix is singular ($\det A = 0$)

1.2.9. p(10) Let $\underline{a}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\underline{a}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $\underline{a}_3 = \begin{pmatrix}1\\3\\2\end{pmatrix}$; we want to show that these are coplanar.

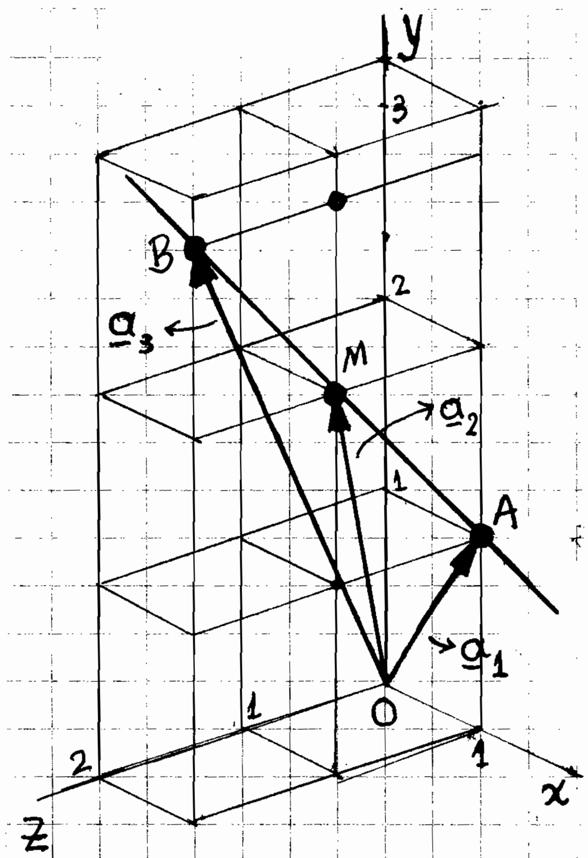


Fig. 2

From the picture,

$\underline{a}_2 = \frac{1}{2}(\underline{a}_1 + \underline{a}_3)$

(M is midpoint of AB in triangle OAB)

In the previous problem we reduced the system $Ax = b$, $A = (\underline{a}_1, \underline{a}_2, \underline{a}_3)$, to echelon form:

$$A \rightarrow U = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 0\end{pmatrix}$$

From this we have:

$$A\underline{n} = 0 \Rightarrow U\underline{n} = 0 \Rightarrow$$

$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1 & 1\\0 & 1\\\cancel{0} & \cancel{0}\end{pmatrix}\begin{pmatrix}n_1\\n_2\end{pmatrix} = -\begin{pmatrix}1\\2\end{pmatrix}n_3$$

$$\Rightarrow n_2 = -2n_3,\ n_1 = -n_2 - n_3 = n_3$$

i.e. $\underline{n} = t\begin{pmatrix}1\\-2\\1\end{pmatrix}$. Then

$$\begin{pmatrix}1\\1\\0\end{pmatrix} - 2\begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}1\\3\\2\end{pmatrix} = \underline{0}$$

The same can be seen by inspection: in Fig. 2 it is easily seen that the three points A, M, B are collinear with M the midpoint of AB.

1.3.1 (p.16)

First, do the algebra:

$$\begin{array}{c} \\ -2 \\ -1 \end{array}\left(\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right) \xrightarrow{(1)} \begin{array}{c} \\ \\ -2 \end{array}\left(\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right) \xrightarrow{(2)} \left(\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right).$$

(1) Elimination: pivots: $(2, 1, -5)$

row operations:

$$\begin{cases} (1a): & \underline{a}^2 \to \underline{a}^2 - 2\underline{a}^1 = \underline{a}^2_{(1)} \\ (1b) & \underline{a}^3 \to \underline{a}^3 - \underline{a}^1 = \underline{a}^3_{(2)} \end{cases}$$

$$(2) \qquad \underline{a}^3_{(2)} \to a^3_{(2)} - \underline{a}^2_{(2)} = \underline{a}^3 - \underline{a}^2 + \underline{a}^1 = \underline{a}^3_{(3)}$$

(2) Back substitution

$$\begin{array}{c} *\frac{1}{2} \\ *1 \\ *-\frac{1}{5} \end{array}\left(\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right) \xrightarrow{(3)} \left(\begin{array}{ccc|c} 1 & -3/2 & 0 & 3/2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)\begin{array}{c} \\ -1 \\ \\ \end{array} \xrightarrow{(4)} \left(\begin{array}{ccc|c} 1 & -3/2 & 0 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)\begin{array}{c} 3/2 \\ \\ \\ \end{array} \xrightarrow{(5)}$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \text{i.e.} \quad (u, v, w) = (3, 1, 0)$$

1.3.3 (p.16) Solve & find pivots:

$$\left.\begin{array}{rl} 2u - v & = 0 \\ -u + 2v - w & = 0 \\ -v + 2w - z & = 0 \\ -w + 2z & = 5 \end{array}\right\}$$

$$\begin{array}{c} \\ \frac{1}{2} \\ \\ \\ \end{array}\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right) \xrightarrow{(1)} \begin{array}{c} \\ \\ 2/3 \\ 0 \end{array}\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & -3/2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right) \xrightarrow{(2)}$$

$$\xrightarrow{(2)} \begin{array}{c} \\ \\ \\ 3/4 \end{array}\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right) \xrightarrow{(3)} \left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right) \Rightarrow$$

~~$z = 5$~~

~~$w = z = 5$~~

~~$\frac{3}{2} v = w \Rightarrow v = \frac{10}{3}$~~

~~$2u = v \Rightarrow u = 5/3$~~

Pivots: $(2, 3/2, 4/3, 5/3)$

$$\Rightarrow z = 4$$

$$w = \frac{3}{4} z = 3$$

$$v = \frac{2}{3} w = 2$$

$$u = \frac{1}{2} v = 1$$

Solution $\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

1.3.11 p.17 Solve $A x_i = b_i$, $i = 1, 2$; here $(A \mid \underline{b}_1 \, \underline{b}_2) =$

$$
\begin{array}{c} \\ -1 \\ -2 \end{array}
\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 6 & 7 \\ 1 & 2 & 2 & 11 & 10 \\ 2 & 3 & -4 & 3 & 3 \end{array}\right)
\xrightarrow{(1)}
\begin{array}{c} \\ \\ -1 \end{array}
\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 6 & 7 \\ 0 & 1 & 1 & 5 & 3 \\ 0 & 1 & -6 & -9 & -11 \end{array}\right)
\xrightarrow{(2)}
\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 6 & 7 \\ 0 & 1 & 1 & 5 & 3 \\ 0 & 0 & -7 & -14 & -14 \end{array}\right)
$$

$$
\xrightarrow{(3)}
\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 6 & 7 \\ 0 & 1 & 1 & 5 & 3 \\ 0 & 0 & 1 & +2 & +2 \end{array}\right)
\begin{array}{c} -1 \\ -1 \\ \\ \end{array}
\xrightarrow{(4)}
\left(\begin{array}{ccc|cc} 1 & 1 & 0 & 4 & 5 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & +2 & +2 \end{array}\right)
\begin{array}{c} -1 \\ \\ \\ \end{array}
$$

$$
\xrightarrow{(5)}
\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & +2 & +2 \end{array}\right)
\quad \text{i.e.} \quad
\underline{x}_2 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \quad
\underline{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}
$$

1.4.4, p.28 Op. count: (1) $A\underline{x}$, $A \in \mathbb{R}^{m \times n}$, $\underline{x} \in \mathbb{R}^n$

(2) AB, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

(1) $$(A\underline{x})_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, m \quad \Rightarrow m * [n \text{ (mult)} + (n-1) \text{ (ad}$$

$$= m(2n-1) \text{ ops}$$

(2) $$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i = 1 \dots m, \; j = 1 \dots p$$

So: $$m * p * [n \text{ (mult)} + (n-1) \text{(add)}] = mp(2n-1) \text{ ops.}$$

1.4.21, p.(29) To show $AB = \sum_{i=1}^{n} \underline{a}_i \underline{b}^i$

with $\underline{a}_i$ = i-th ~~row~~ column of A

$\underline{b}^i$ = i-th row of B:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

while $(\underline{a}_k \underline{b}^k)_{ij} = a_{ik} b_{kj}$ (no sum)

Summing over k: $\sum_{k=1}^{n} \underline{a}_k \underline{b}^k = \sum_{k=1}^{n} a_{ik} b_{kj} = (AB)_{ij}$

and the two expressions are equal since they have equal components.