The Sherman-Morrison-Woodburry formula and its uses

1. Rank 1 modifications of the identity

Let $T = I + uv^T$; look for T^{-1} in form:

$T^{-1} = I + \sigma u v^T$. Indeed

$$(I+uv^T)(I+\sigma uv^T) = I + (1+\sigma)uv^T + u[\sigma v^T u]v^T$$

$$+ \quad = 0$$

i.e. $1+\sigma+\sigma v^T u = 0$ is the required condition

Lemma ~~Theorem~~ 1: $(I+uv^T)^{-1} = I + \sigma uv^T$ if $\sigma = -(1+v^T u)^{-1}$

2. Rank 1 mod. of an invertible matrix A.

Now consider $S = A + uv^T$. To find inverse:

$$S^{-1} = (A+uv^T)^{-1} = [A(I+A^{-1}uv^T)]^{-1} = (I+A^{-1}uv^T)^{-1}A^{-1}$$

Introduce $w = A^{-1}u \Leftrightarrow Aw = u$. Then, using Thm. 1:

$$S^{-1} = (I+wv^T)^{-1}A^{-1} = (I+\sigma wv^T)A^{-1}$$

$$= \cancel{A^{-1}}(I+\sigma A^{-1}uv^T)A^{-1}$$

$$= A^{-1} - \sigma A^{-1}uv^T A^{-1}$$

with $\sigma = (1+v^T w)^{-1} = (1+v^T A^{-1}u)^{-1}$

Written in symmetric form; S becomes:

Lemma ~~Theorem~~ 2: If $S = A+uv^T$; A^{-1} exists.

$$S^{-1} = A^{-1} - A^{-1}u\sigma v^T A^{-1} \quad ; \quad \sigma = (1+v^T A^{-1}u)^{-1}$$

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3. Rank m modification of identity

Let $R = I + UV^T$; R $n\times n$; $U, V : n \times m$.
(obviously $m \leq n$ if U, V are rank m). ~~Again,~~ Now, look for a matrix $T_{m\times m}$ so that

$$R^{-1} = I + UTV^T :$$

$$(I + UV^T)(I + UTV^T) = I + UV^T + UTV^T + U(V^TUT)V^T$$

$$= I + U\{I + T + V^TUT\}V^T$$

$$= I + U\{I + (I + V^TU)T\}V^T = I$$

Provided that $T = -(I + V^TU)^{-1}$

~~Theo~~ Lemma 3: Given $n\times n$ matrix $I + UV^T$ where U, V are $n \times m$ $(m \leq n)$. Then $(I + UV^T)^{-1} = I + UT^{-1}V^T$ where $T = (I + V^TU)^{m n}$.

4. The SMW formula

$A^{n\times n}$, $U, V^{n\times m}$. Then $B = A + UV^T$ has inverse $(A + UV^T)^{-1} =$

$$= [A(I + A^{-1}UV^T)]^{-1} = (I + A^{-1}UV^T)^{-1}A^{-1}.$$ ~~Since~~

Introduce $W = A^{-1}U$ (W satisfies: $AW = U$)

so that $(I + WV^T)^{-1} = I - WT^{-1}V^T$ by Lemma 3,

where $T = (I + V^TW) = I + V^TA^{-1}U$. Then

$$B^{-1} = (I - WT^{-1}V^T)A^{-1} = A^{-1} - A^{-1}UT^{-1}V^TA^{-1}.$$

Theorem: $(A + UV^T)^{-1} = A^{-1} - A^{-1}UT^{-1}V^TA^{-1}$; $T = I + V^TA^{-1}U$

Uses: Suppose we have constructed the LU factorization of a $n\times n$ matrix A: $A = LU$ but we need to modify some of the entries of A. ~~For example~~ ~~This happens, for example,~~ ~~if we are solving~~ For example to alter the (i,j) entry $a_{ij} \to a_{ij} + \delta$ we need $A + \delta e_i e_j^T$; to alter a set $(i_\ell, k_\ell)_{\ell=1,\dots,m}$ of entries we need $B = A + \sum_{\ell=1}^{m} \delta_\ell e_{i_\ell} e_{j_\ell}^T$

$\equiv A + UV^T$ where we used $U_\ell = e_{i_\ell}$, $V_\ell = \delta_\ell e_{j_\ell}$.

Then we can use the SMW formula to ~~construct~~ ~~the inverse~~ ~~LU factorization~~ of B with less than n^3 operations as follows: We know how to find $z = A^{-1}b$ (solve $Az = b$).

To solve $(A + UV^T)x = b$

$$x = (A^{-1}b) + A^{-1}UT^{-1}V^T(A^{-1}b)$$

$$= z + (A^{-1}U)T^{-1}V^T z$$

We also know how to find $W = A^{-1}U$ (solve $AW = U$).

$$x = z + WT^{-1}V^T z$$

i.e. we have the solution of $(A + UV^T)x = b$ $(m+1)n^2$ ops

By solving the $m+1$ problems $Aw_i = u_i$; $i = 1,\dots,m$ and $Az = b$, and computing the inverse of the $m\times m$ matrix $T = I + V^T A^{-1} U = I + V^T W$ (m^3 ops)

Question: for what values of m is this method better than simply computing the LU of B and solving for x?