Matrix Theory, Fall 2009
Midterm 1

October 22, 2009

Instructions: There are six (6) questions on this examination.
Grads: work problems **1,2,3** and any **two** of problems **4,5,6**
Undergrads: work problems **1,2,3** and any **one** of problems **4,5,6**
UG's get 20 bonus points. The maximum points possible are 100

1. (15 points)
   (i) Solve by elimination and back substitution:

$$\begin{array}{rcrcrcl} u & & & + & w & = & 4 \\ u & + & v & & & = & 3 \\ u & + & v & + & w & = & 6 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} & & v & + & w & = & 0 \\ u & & & + & w & = & 0 \\ u & + & v & & & = & 6 \end{array}$$

   (ii) Factor the above matrices into $A = LU$ or $PA = LU$.

2. (15 points)
(i) If A is square, show that the nullspace of A^2 contains the nullspace of A.
(ii) Show also that the column space of A^2 is contained in the column space of A.
(iii)If e_1,e_2,e_3 are in the column space of a 3×5 matrix, does it have a right inverse?

3. (30 points) Suppose the matrices in $PA = LU$ are:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 4 & 2 & 1 \\ 4 & -2 & 9 & 1 & 4 \\ 2 & -1 & 5 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) What is the rank of A?
(ii) What is a basis for $\mathcal{C}(A^T)$?
(iii) TRUE or FALSE: Rows 1, 2, 3 of A are linearly independent.
(iv) What is a basis for $\mathcal{C}(A)$?
(v) What is the dimension of $\mathcal{N}(A^T)$?
(vi) What is the general solution to $Ax = 0$?

4. (20 points)
The Sherman-Morrison-Woodbury formula:
(i) Let A be $n \times n$, nonsingular. Show that if the $m \times m$ matrix T defined by

$$T = I + V^T A^{-1} U$$

is nonsingular, then

$$(A + UV^T)^{-1} = A^{-1} - A^{-1} U T^{-1} V^T A^{-1}.$$

What are the dimensions of the matrices V and U?
(ii) If the assumptions of part (i) are satisfied, what are the relative sizes of m and n (i.e. for all formulas above to make sense can we have $m < n$ or $m > n$ or $m = n$)?

5. (20 points)
Consider the overdetermined problem

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Find the Least Squares solution by using the QR factorization.

6. (20 points) If A is a 12×7 incidence matrix from a connected graph, what is its rank? How many free variables in the solution to $Ax = b$? How many free variables in the solution to $A^T y = f$? How many edges must be removed to leave a spanning tree?