Solutions, 316-XVI

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1 Problem 7.4.7

Determine the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\}$$

Solution:

$$\begin{aligned}
\frac{2s+16}{s^2+4s+13} &= \frac{2s+16}{(s^2+4s+4)-4+13} \\
&= \frac{2(s+2)-4+16}{(s+2)^2+9} \\
&= \frac{2(s+2)+4(3)}{(s+2)^2+3^2}
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+3^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} \\
&= 2e^{-2t}\cos 3t + 4e^{-2t}\sin 3t
\end{aligned}$$

2 Problem 7.4.9

Determine the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{3s-15}{2s^2-4s+10}\right\}$$

Solution:

$$\begin{aligned}
\frac{3s-15}{2s^2-4s+10} &= \frac{3s-15}{2((s^2-2s+1)-1+5)} \\
&= \frac{3(s-1)+3-15}{2((s-1)^2+4)} \\
&= \frac{3(s-1)-6(2)}{2((s-1)^2+2^2)}
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3s-15}{2s^2-4s+10}\right\} &= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+2^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\} \\
&= \frac{3}{2}e^{-t}\cos 2t - 3e^{-t}\sin 2t
\end{aligned}$$

3 Problem 7.4.11

Determine the partial fractions expansion for

$$F(s) = \frac{s^2-26s-47}{(s-1)(s+2)(s+5)} \ .$$

Solution:

$$\begin{aligned}
F(s) &= \frac{s^2-26s-47}{(s-1)(s+2)(s+5)} \\
&= \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+5}
\end{aligned}$$

Then

$$\begin{aligned}
(s-1)F(s) &= \frac{s^2-26s-47}{(s+2)(s+5)} \\
&= A + (s-1)\frac{B}{s+2} + (s-1)\frac{C}{s+5} \Rightarrow \\
\lim_{s\to 1}(s-1)F(s) &= \left.\frac{s^2-26s-47}{(s+2)(s+5)}\right|_{s=1} = \frac{1-26-47}{(1+2)(1+5)} = \frac{-72}{18} = -4 = A
\end{aligned}$$

also

$$\begin{aligned}
(s+2)F(s) &= \frac{s^2-26s-47}{(s-1)(s+5)} \\
&= (s+2)\frac{-4}{s-1} + B + (s+2)\frac{C}{s+5} \Rightarrow \\
\lim_{s\to -2}(s+2)F(s) &= \left.\frac{s^2-26s-47}{(s-1)(s+5)}\right|_{s=-2} = \frac{4+52-47}{(-2-1)(-2+5)} = \frac{9}{-9} = -1 = B
\end{aligned}$$

and

$$\begin{aligned}
(s+5)F(s) &= \frac{s^2-26s-47}{(s-1)(s+2)} \\
&= (s+5)\frac{-4}{s-1} + (s+5)\frac{B}{s+2} + C \Rightarrow \\
\lim_{s\to -5}(s+5)F(s) &= \left.\frac{s^2-26s-47}{(s-1)(s+2)}\right|_{s=-5} = \frac{25+130-47}{(-5-1)(-5+2)} = \frac{108}{18} = 6 = B
\end{aligned}$$

so that

$$F(s) = -4\frac{1}{s-1} - \frac{1}{s+2} + 6\frac{1}{s+5}$$

and

$$\mathcal{L}^{-1}\left\{F(s)\right\} = -4e^{t} - e^{-2t} + 6e^{-5t} \; .$$

4 Problem 7.4.15

Determine the partial fractions expansion for

$$F(s) = \frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} \; .$$

Solution:

$$\begin{aligned}
F(s) &= \frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} \\
&= \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+1+4} \\
&= \frac{A}{s+1} + \frac{Bs+C}{(s-1)^2+4}
\end{aligned}$$

Then

$$\begin{aligned}
(s+1)F(s) &= \frac{-2s^2+8s-14}{(s^2-2s+5)} \\
&= A + (s+1)\frac{(Bs+C)}{s^2-2s+1+4} \Rightarrow \\
\lim_{s\to -1}(s+1)F(s) &= \left.\frac{-2s^2+8s-14}{(s^2-2s+5)}\right|_{s=-1} = \frac{-2-8-14}{(1+2+5)} = \frac{-24}{8} = 3 = A
\end{aligned}$$

Subtracting:

$$\begin{aligned}
\frac{Bs+C}{(s-1)^2+4} &= \frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} - \frac{-3}{(s+1)} \\
&= \frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} + \frac{3(s^2-2s+5)}{(s+1)(s^2-2s+5)} \\
&= \frac{-2s^2+8s-14+3(s^2-2s+5)}{(s+1)(s^2-2s+5)} \\
&= \frac{s^2+2s+1}{(s+1)(s^2-2s+5)} \\
&= \frac{(s+1)^2}{(s+1)(s^2-2s+5)} \\
&= \frac{s+1}{(s^2-2s+5)} \\
&= \frac{(s-1)+2}{(s-1)^2+4} \\
&= \frac{(s-1)}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}
\end{aligned}$$

so that

$$F(s) = -3\frac{1}{s+1} + \frac{(s-1)}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}$$

and

$$\mathcal{L}^{-1}\left\{F(s)\right\} = -3e^{-t} + e^t\cos 2t + e^t\sin 2t \ .$$

5 Problem 7.4.23

Determine the inverse Laplace transform

$$F(s) = \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)} \ .$$

Solution:

$$\begin{aligned} F(s) &= \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)} \\ &= \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{C}{s+1} \end{aligned}$$

Then

$$A = \lim_{s \to -3} (s+3)^2 F(s) = \left. \frac{5s^2 + 34s + 53}{s+1} \right|_{s=-3} = \frac{-4}{-2} = 2$$

and

$$C = \lim_{s \to -1} (s+1) F(s) = \left. \frac{5s^2 + 34s + 53}{(s+3)^2} \right|_{s=-1} = \frac{24}{4} = 6$$

while

$$B = \lim_{s \to -3} \frac{d}{ds} \left((s+3)^2 F(s) \right) = \left(\frac{10s + 34}{s+1} - \frac{5s^2 + 34s + 53}{(s+1)^2} \right)_{s=-3} = \frac{4}{-2} - \frac{-4}{4} = -1$$

Then

$$\begin{aligned} F(s) &= \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} \\ &= \frac{2}{(s+3)^2} - \frac{1}{s+3} + \frac{6}{s+1} \end{aligned}$$

and

$$f(t) = 2te^{-3t} - e^{-3t} + 6e^{-t} \ .$$

6 Problem 7.4.26

Determine the inverse Laplace transform

$$F(s) = \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} \ .$$

Solution:

$$\begin{aligned} F(s) &= \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} \\ &= \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-2} \end{aligned}$$

Then

$$A = \lim_{s\to 0} s^3 F(s) = \left.\frac{7s^3 - 2s^2 - 3s + 6}{s-2}\right|_{s=0} = \frac{6}{-2} = -3$$

and

$$D = \lim_{s\to 2} (s-2) F(s) = \left.\frac{7s^3 - 2s^2 - 3s + 6}{s^3}\right|_{s=2} = \frac{48}{8} = 6$$

while

$$B = \lim_{s\to 0} \frac{d}{ds}\left(s^3 F(s)\right) = \left(\frac{21s^2 - 4s - 3}{s-2} - \frac{7s^3 - 2s^2 - 3s + 6}{(s-2)^2}\right)_{s=0} = \frac{3}{2} - \frac{6}{4} = 0$$

and

$$\begin{aligned} C &= \lim_{s\to 0} \frac{1}{2}\frac{d^2}{ds^2}\left(s^3 F(s)\right) \\ &= \frac{1}{2}\frac{d}{ds}\left(\frac{21s^2 - 4s - 3}{s-2} - \frac{7s^3 - 2s^2 - 3s + 6}{(s-2)^2}\right)_{s=0} \\ &= \frac{1}{2}\left(\frac{42s - 4}{s-2} - \frac{21s^2 - 4s - 3}{(s-2)^2} - \frac{21s^2 - 4s - 3}{(s-2)^2} + 2\frac{7s^3 - 2s^2 - 3s + 6}{(s-2)^3}\right)_{s=0} \\ &= \frac{1}{2}\left(\frac{-4}{-2} - \frac{-3}{(-2)^2} - \frac{-3}{(-2)^2} + 2\frac{6}{(-2)^3} = 2 + \frac{3}{4} + \frac{3}{4} - \frac{3}{2}\right) = 1 \end{aligned}$$

Then

$$\begin{aligned} F(s) &= \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} \\ &= \frac{-3}{s^3} + \frac{1}{s} + \frac{6}{s-2} \end{aligned}$$

and

$$f(t) = -\frac{3}{2}t^2 + 1 + 6e^{2t} \ .$$

7 Problem 7.5.9

Solve the given IVP using Laplace transforms

$$z'' + 5z' - 6z = 21e^t \ , \ z(0) = -1 \ , \ z'(0) = 9 \ .$$

Solution:

$$\begin{aligned}
\mathcal{L}\{z'' + 5z' - 6z\} &= \mathcal{L}\left\{21e^t\right\} = \frac{21}{s-1} \\
\left(s^2 + 5s - 6\right) Z(s) &= (s+5)z(0) + z'(0) + \frac{21}{s-1} \\
Z(s) &= \frac{-(s+5)+9}{(s+6)(s-1)} + \frac{21}{(s-1)^2(s+6)} \\
Z(s) &= \frac{(-s+4)(s-1)}{(s+6)(s-1)^2} + \frac{21}{(s-1)^2(s+6)} = \frac{-s^2+5s+17}{(s-1)^2(s+6)} \\
Z(s) &= \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+6}
\end{aligned}$$

Then

$$A = \lim_{s\to 1}(s-1)^2 Z(s) = \left.\frac{-s^2+5s+17}{s+6}\right|_{s=1} = \frac{21}{7} = 3$$

and

$$C = \lim_{s\to -6}(s+6)F(s) = \left.\frac{-s^2+5s+17}{(s-1)^2}\right|_{s=-6} = \frac{-49}{49} = -1$$

while

$$B = \lim_{s\to 1}\frac{d}{ds}\left((s-1)^2F(s)\right) = \left(\frac{-2s+5}{s+6} - \frac{-s^2+5s+17}{(s+6)^2}\right)_{s=1} = \frac{3}{7} - \frac{21}{49} = 0$$

Then

$$\begin{aligned}
Z(s) &= \frac{-s^2+5s+17}{(s-1)^2(s+6)} \\
&= \frac{3}{(s-1)^2} - \frac{1}{s+6}
\end{aligned}$$

and

$$z(t) = 3te^{t} - e^{-6t} \ .$$

8 Problem 7.5.13

Solve the given IVP using Laplace transforms

$$y'' - y' - 2y = -8\cos t - 2\sin t \ ; \ y(\pi/2) = 1 \ , \ y'(\pi/2) = 0 \ .$$

Solution:
We first introduce the new independent variable

$$\tau := t - \pi/2$$

and define

$$x(\tau) = x(t - \pi/2) = y(t)$$

so that

$$\frac{dx}{d\tau} = \frac{dy}{dt} \ , \ \frac{d^2x}{d\tau^2} = \frac{d^2y}{dt^2}$$

and

$$x(0) = y(\pi/2) = 1 \ , \ x'(0) = y'(\pi/2) = 0 \ .$$

Then $x(\tau)$ satisfies the IVP:

$$x'' - x' - 2x = -8\cos(\tau + \pi/2) - 2\sin(\tau + \pi/2) = 8\sin\tau - 2\cos\tau \ ,$$

$$x(0) = 1 \ , \ x'(0) = 0 \ .$$

We now proceed as usual with the Laplace transform (where we now write t instead of τ just to keep things looking familiar! At the end we will need to fix this!):

$$\begin{aligned}
\mathcal{L}\{x'' - x' - 2x\} &= \mathcal{L}\{8\sin t - 2\cos t\} \\
\left(s^2 - s - 2\right) Z(s) &= (s-1)z(0) + z'(0) + \frac{-2s+8}{s^2+1} \\
X(s) &= \frac{s-1}{(s+1)(s-2)} + \frac{-2s+8}{(s+1)(s-2)(s^2+1)} \\
&= \frac{(s^2+1)(s-1)}{(s+1)(s-2)(s^2+1)} + \frac{-2s+8}{(s+1)(s-2)(s^2+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s^3 - s^2 + s - 1 - 2s + 8}{(s+1)(s-2)(s^2+1)} \\
&= \frac{s^3 - s^2 - s + 7}{(s+1)(s-2)(s^2+1)} \\
&= \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}
\end{aligned}
$$

Then

$$
A = \lim_{s \to -1} (s+1) X(s) = \left. \frac{s^3 - s^2 - s + 7}{(s-2)(s^2+1)} \right|_{s=-1} = \frac{6}{-6} = -1
$$

and

$$
B = \lim_{s \to 2} (s-2) X(s) = \left. \frac{s^3 - s^2 - s + 7}{(s+1)(s^2+1)} \right|_{s=2} = \frac{9}{15} = \frac{3}{5}
$$

Then

$$
\begin{aligned}
\frac{s^3 - s^2 - s + 7}{(s+1)(s-2)(s^2+1)} &= \frac{-1}{s+1} + \frac{3}{5(s-2)} + \frac{Cs+D}{s^2+1} \Rightarrow \\
\frac{s^3 - s^2 - s + 7}{s^2+1} &= -(s-2) + \frac{3}{5}(s+1) + (s+1)(s-2)\frac{Cs+D}{s^2+1}
\end{aligned}
$$

where we multiplied both sides by $(s+1)(s-2)$. Now we set s equal to some convenient values to get a system for C, D.

1. $s = 0$:

$$
7 = 2 + \frac{3}{5} + 1(-2)D \Rightarrow D = \frac{-11}{5}
$$

2. $s = 1$:

$$
\frac{1 - 1 - 1 + 7}{1+1} = -(-1) + \frac{6}{5} + 2(-1)\frac{C+D}{2} \Rightarrow C = -\frac{4}{5} + \frac{11}{5} = \frac{7}{5}
$$

and finally

$$
X(s) = \frac{-1}{s+1} + \frac{3}{5(s-2)} + \frac{7s - 11}{5(s^2+1)}
$$

so that

$$
x(t) = -e^{-t} + \frac{3}{5}e^{2t} + \frac{7}{5}\cos t - \frac{11}{5}\sin t \ ,
$$

and rewriting in terms of $y(t)$, that is changing from t to τ in the above expression for x we have

$$
y(t) = -e^{-t+\pi/2} + \frac{3}{5}e^{2t+\pi} + \frac{7}{5}\sin t + \frac{11}{5}\cos t \ .
$$

9 Problem 7.5.27

Solve the given 3rd order IVP using Laplace transforms

$$y''' + 3y'' + 3y' + y = 0 \ ; \ y(0) = -4 \ , \ y'(0) = 4 \ , \ y''(0) = -2 \ .$$

Solution:

$$\begin{aligned}
\mathcal{L}\left\{y''' + 3y''(0) + 3y' + y\right\} &= 0 \\
\left(s^3 + 3s^2 + 3s + 1\right) Y(s) &= (s^2 + 3s + 3)y(0) + (s + 3)y'(0) + y'' \\
&= -4(s^2 + 3s + 3) + 4(s + 3) - 2 \\
&= -4s^2 - 8s - 2 \Rightarrow \\
Y(s) &= \frac{-4s^2 - 8s - 2}{s^3 + 3s^2 + 3s + 1} \\
&= \frac{-4(s + 1)^2 + 2}{(s + 1)^3} = -\frac{4}{s + 1} + \frac{2}{(s + 1)^3} \rightarrow \\
y(t) &= (t^2 - 4)e^{-t} \ .
\end{aligned}$$

	t-domain $(f(t))$	s-domain $(F(s))$
1	$f(t)$	$F(s)$
2	$C_1f_1(t)+C_2f_2(t)$	$C_1F_1(s)+C_2F_2(s)$
3	1	$\frac{1}{s}$
4	t	$\frac{1}{s^2}$
5	t^n	$\frac{n!}{s^{n+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$e^{at}f(t)$	$F(s-a)$
8	$\cos bt$	$\frac{s}{s^2+b^2}$
9	$\sin bt$	$\frac{b}{s^2+b^2}$
10	$f'(t)$	$sF(s)-f(0)$
11	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
12	$tf(t)$	$-F'(s)$
13	$t^nf(t)$	$(-1)^nF^{(n)}(s)$
14	$\int_0^t g(\tau)h(t-\tau)d\tau$	$G(s)H(s)$
15	$\int_0^t g(\tau)d\tau$	$\frac{1}{s}G(s)$
16	$f(t-a)u(t-a)$	$e^{-as}F(s)$
17	$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
18	$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$

Table 1: *Useful Laplace transforms*