Solutions, 316-XV

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1 Problem 7.2.15

Determine the transform

$$\mathcal{L}\left\{t^3 - te^t + e^{4t}\cos t\right\}$$

Solution:

$$\begin{aligned}
\mathcal{L}\left\{t^3 - te^t + e^{4t}\cos t\right\} &= \mathcal{L}\left\{t^3\right\} - \mathcal{L}\left\{te^t\right\} + \mathcal{L}\left\{e^{4t}\cos t\right\} \\
&= \frac{3!}{s^{3+1}} - \frac{1}{(s-1)^{1+1}} + \frac{s-4}{(s-4)^2+1} \\
&= \frac{6}{s^4} - \frac{1}{s^2-2s+1} + \frac{s-4}{s^2-8s+17}
\end{aligned}$$

2 Problem 7.3.5

Determine the transform

$$\mathcal{L}\left\{2t^2e^{-t} - t + \cos 4t\right\}$$

Solution:

$$\begin{aligned}
\mathcal{L}\left\{2t^2e^{-t} - t + \cos 4t\right\} &= 2\mathcal{L}\left\{t^2e^{-t}\right\} - \mathcal{L}\left\{t\right\} + \mathcal{L}\left\{\cos 4t\right\} \\
&= 2\frac{2!}{(s+1)^{2+1}} - \frac{1!}{s^{1+1}} + \frac{s}{s^2+4^2} \\
&= \frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+16}
\end{aligned}$$

3 Problem 7.3.19

Determine the transform

$$\mathcal{L}\{\cos nt \sin mt\} \quad , \quad m \neq n$$

Solution:
Using the familiar trigonometric identity:

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
2 \sin A \cos B &= \sin(A - B) + \sin(A + B) \\
2 \cos A \sin B &= \sin(A + B) - \sin(A - B)
\end{aligned}$$

so that, if we set

$$A = nt \ , \ B = mt$$

i.e.

$$A + B = (n + m)t \ , \ A - B = (n - m)t$$

we have

$$\cos nt \sin mt = \frac{1}{2} \sin(n + m)\, t - \frac{1}{2} \sin(n - m)\, t$$

Applying these identities to our problem:

$$\begin{aligned}
\mathcal{L}\{\cos nt \sin mt\} &= \frac{1}{2}\mathcal{L}\{\sin(n + m)\, t\} - \frac{1}{2}\mathcal{L}\{\sin(n - m)\, t\} \\
&= \frac{1}{2} \frac{n + m}{s^2 + (n + m)^2} - \frac{1}{2} \frac{n - m}{s^2 + (n - m)^2}
\end{aligned}$$

4 Problem 7.3.25

Use the formula

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n F(s)}{ds^n}(s) \ .$$

to help determine:

1. $\mathcal{L}\left\{t \cos bt\right\}$
2. $\mathcal{L}\left\{t^2 \cos bt\right\}$

Solution:
Since

$$F(s) := \mathcal{L}\left\{\cos bt\right\} = \frac{s}{s^2 + b^2}$$

we have

$$\begin{aligned}
\mathcal{L}\left\{t \cos bt\right\} &= -\frac{dF(s)}{ds} \\
&= -\frac{d}{ds}\frac{s}{s^2+b^2} \\
&= -\frac{1}{s^2+b^2} + \frac{2s^2}{\left(s^2+b^2\right)^2} \\
&= -\frac{(s^2+b^2) - 2s^2}{\left(s^2+b^2\right)^2} \\
&= \frac{s^2-b^2}{\left(s^2+b^2\right)^2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\left\{t^2 \cos bt\right\} &= \frac{d^2F(s)}{ds^2} \\
&= -\frac{d}{ds}\frac{s^2-b^2}{\left(s^2+b^2\right)^2} \\
&= -\frac{2s}{\left(s^2+b^2\right)^2} + \frac{4s(s^2-b^2)}{\left(s^2+b^2\right)^3} \\
&= -\frac{2s\left(s^2+b^2\right)}{\left(s^2+b^2\right)^3} + \frac{4s(s^2-b^2)}{\left(s^2+b^2\right)^3} \\
&= \frac{2s(s^2-3b^2)}{\left(s^2+b^2\right)^3}
\end{aligned}$$

	t-domain ($f(t)$)	s-domain ($F(s)$)
1	$f(t)$	$F(s)$
2	$C_1 f_1(t) + C_2 f_2(t)$	$C_1 F_1(s) + C_2 F_2(s)$
3	1	$\frac{1}{s}$
4	t	$\frac{1}{s^2}$
5	t^n	$\frac{n!}{s^{n+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$e^{at} f(t)$	$F(s-a)$
8	$\cos bt$	$\frac{s}{s^2+b^2}$
9	$\sin bt$	$\frac{b}{s^2+b^2}$
10	$f'(t)$	$sF(s) - f(0)$
11	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
12	$tf(t)$	$-F'(s)$
13	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
14	$\int_0^t g(\tau) h(t-\tau) d\tau$	$G(s)H(s)$
15	$\int_0^t g(\tau) d\tau$	$\frac{1}{s} G(s)$
16	$f(t-a)u(t-a)$	$e^{-as} F(s)$
17	$t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
18	$t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$

Table 1: *Useful Laplace transforms*