(1) (20 pts) Find the general solution of the inhomogeneous differential equation (use any method)

$$y'' + 4y = t\cos t\ .$$

Char. eqn: $r^2 + 4r = 0 \Rightarrow r = \pm 2i$: $t\cos t$ not resonant

$y_h = C_1\cos 2t + C_2\sin 2t$

Try $y_p = (At+B)\cos t + (Ct+D)\sin t$

$y_p' = [Ct + (A+D)]\cos t + [(-A)t + (-B+C)]\sin t$

$y_p'' = [-At + (-B+2C)]\cos t + [-Ct + (-2A-D)]\sin t$

$y_p'' + 4y = \begin{bmatrix} -At - B + 2C \\ +4At + 4B \end{bmatrix}\cos t + \begin{bmatrix} -Ct - 2A - D \\ +4Ct + 4D \end{bmatrix}\sin t = t\cos t$

$\Rightarrow \{3At + 3B + 2C\}\cos t + \{3Ct - 2A + 3D\}\sin t = t\cos t$

$3A = 1 \Rightarrow A = 1/3 \quad 3B + 2C = 0 \Rightarrow B = 0$

$3C = 0 \quad -2A + 3D = 0 \Rightarrow D = \frac{2}{3}A = \frac{2}{9}$

i.e. $\boxed{y_p = \frac{1}{3}t\cos t + \frac{2}{9}\sin t}$

$y_g = C_1\cos 2t + C_2\sin 2t + \frac{1}{3}t\cos t + \frac{2}{9}\sin t$

Check: $y_p' = \frac{1}{3}\cos t - \frac{1}{3}t\sin t + \frac{2}{9}\cos t$

$y_p'' = -\frac{1}{3}\sin t - \frac{1}{3}\sin t - \frac{1}{3}t\cos t - \frac{2}{9}\sin t$ $\Big\}$ $t\cos t$ $\left(\frac{8}{9} - \cancel{\frac{2}{9} - \frac{3}{9}} \frac{3}{9}\right)\sin t$

$+$

$4y = \frac{4}{3}t\cos t + \frac{8}{9}\sin t$

(2) (20 pts) Find the general solution of the inhomogeneous differential equation (use any method)

$$y'' - 5y' + 6y = te^{2t}.$$

$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0$

$y_h = c_1 e^{2t} + c_2 e^{3t}$

Try $y_p = t(At+B)e^{2t} = (At^2 + Bt)e^{2t}$

$y_p' = [(2At+B) + 2(At^2+Bt)]e^{2t} = [2At^2 + 2(A+B)t + B]e^{2t}$

$y_p'' = [4At + 2(A+B)] + 2[2At^2 + 2(A+B)t + B] =$

$= [4At^2 + (8A+4B)t + (2A+4B)]e^{2t}$

$-5y_p'$ $\quad [-10At^2 - 10(A+B)t - 5B]\, e^{2t}$

$+6y$ $\quad (6At^2 + 6Bt)\, e^{2t}$

$[0t^2 + (-2A)t + (2A-B)]e^{2t} = te^{2t}$

$\Rightarrow \left.\begin{aligned} -2A &= 1 \Rightarrow A = -1/2 \\ 2A - B &= 0 \Rightarrow B = 2A = -1 \end{aligned}\right\}$

$y_p = \left(-\frac{t^2}{2} - t\right)e^{2t}$

3

$y_g = c_1 e^{2t} + c_2 e^{3t} - \frac{t}{2}(t+2)e^{2t}$

$y_p' = (-t-1)e^{2t} + (-t^2-2t)e^{2t} = (-t^2-3t-1)e^{2t}$

$y_p'' = (-2t-3) + 2(-t^2-3t-1) = (-2t^2-8t-5)e^{2t}$

$(-2t^2-8t-5)$
$(+5t^2+15t+5)$
$(-3t^2-6t)$

$(0t^2 + t + 0)\, e^{2t}$

(3) (20 pts) Give the inverse Laplace transforms of the following functions (where applicable, use the partial fractions expansion expression without evaluating the coefficients):

1.

$$\frac{3}{s(s-1)^2(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{s+1}$$

$$\Rightarrow f = A + Be^{t} + Cte^{t} + De^{-t}$$

2.

$$\frac{s}{(s^2+1)^2} \longleftarrow \cancel{\frac{s^2}{(s^2+1)}} = \frac{s^2+1-1}{s^2+1} = 1 - \frac{1}{s^2+1} =$$

$$\frac{s}{(s^2+1)^2} = \frac{1}{2}\,\frac{2s}{(s^2+1)^2} \rightarrow \frac{1}{2}\,t\sin t$$

3.

$$\frac{1}{s^2+4s+5} = \frac{1}{(s+2)^2+1} \rightarrow e^{-2t}\sin t$$

4.

$$\frac{1}{(s+1)(s^2+2s+10)} = \frac{1}{(s+1)\left[(s+1)^2+3^2\right]}$$

$$= \frac{A}{s+1} + \frac{B(s+1)}{(s+1)^2+3^2} + \frac{3C}{(s+1)^2+3^2}$$

$$\longrightarrow Ae^{-t} + Be^{-t}\cos 3t + Ce^{-t}\sin 3t$$

(4) (20 pts) Solve the initial value problem using Laplace transforms

$$y'' - 3y' - 4y = e^{-t} \; ; \; y(0) = 0 \; ; \; y'(0) = 0 \, .$$

$$(s^2 - 3s - 4)Y = \frac{1}{s+1} \Rightarrow (s-4)(s+1)$$

$$Y = \frac{1}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$$

$$C = \left.\frac{1}{(s+1)^2}\right|_{s=4} = \frac{1}{25}$$

$$B = \left.\frac{1}{s-4}\right|_{s=-1} = \frac{1}{-5}$$

$$A = \left.\frac{d}{ds}\left(\frac{1}{s-4}\right)\right|_{s=-1} = -\frac{1}{(s-4)^2} = -\frac{1}{25}$$

$$Y(s) = -\frac{1}{25}\frac{1}{s+1} - \frac{1}{5}\frac{1}{(s+1)^2} + \frac{1}{25}\frac{1}{s-4}$$

$$y(t) = -\frac{1}{25}e^{-t} - \frac{1}{5}e^{-t}t + \frac{1}{25}e^{4t}$$

(5) (20 pts) Determine all critical points. Find the corresponding linear system near each critical point. Find the eigenvalues and draw conclusions about type and stability of each critical point. Draw as much of the phase portrait as you can, incorporating all of the above information.

$$\frac{dx}{dt} = x + y = F = 0 \Rightarrow x = -y$$
$$\frac{dy}{dt} = x + 2y - xy - 2 = G = 0 \Rightarrow y + y^2 - 2 = 0$$

$(y+2)(y-1) = 0$

$y = 1,\ x = -1$

$y = -2,\ x = 2$

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1-y & 2-x \end{pmatrix}$$

$(1,-1)$: $\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{(-1,1)} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ $\quad \lambda_1 = 1,\ \lambda_2 = 3$: unstable node

$\begin{pmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow$ $(1-\lambda)a + b = 0$: $a = 1,\ b = 1-\lambda = \begin{cases} 0 \\ -2 \end{cases}$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t}$$

$(2,-2)$: $\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$

$\det \begin{pmatrix} 1-\lambda & 1 \\ 3 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 3 = 0$

$\lambda = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}$

unstable saddle

$\begin{pmatrix} 1-\lambda & 1 \\ 3 & -\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$: $3a - \lambda b = 0$

$a = \lambda,\ b = 3$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = c_1 \begin{pmatrix} \frac{1+\sqrt{13}}{2} \\ 3 \end{pmatrix} e^{\left(\frac{1+\sqrt{13}}{2}\right)t} + c_2 \begin{pmatrix} \frac{1-\sqrt{13}}{2} \\ 3 \end{pmatrix} e^{\left(\frac{1-\sqrt{13}}{2}\right)t}$$