10.1/8

$$(5)\ I=\int_{-\infty}^{\infty} e^{iax} f(x)\,dx \quad, a>0$$

with $|f(z)| < \dfrac{M}{|z|}$, for $|z| > R_0$

$$I = \lim_{R\to\infty}\left(\int_{-R}^{R} + \int_{C_R}\right) = 2\pi i \sum \left(\begin{matrix}\text{Res}\\ \text{in upper}\\ \text{half-plane}\end{matrix}\right)$$

(if $a<0$, complete in lower half).

<u>Jordan lemma</u>: $\left|\int_{C_R} f e^{iaz} dz\right| \xrightarrow{R\to\infty} 0$

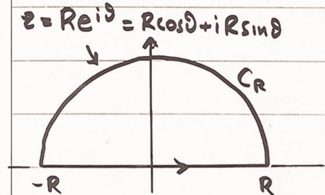
$$\left|\int_{C_R}\cdots\right| \le \int_{C_R} |f(Re^{i\theta})|\cdot e^{-aR\sin\theta} R\,d\theta < M\int_0^{\pi} e^{-aR\sin\theta} d\theta$$

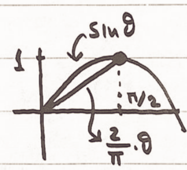
$$= 2M\int_0^{\pi/2} e^{-aR\sin\theta} d\theta$$

$$\le 2M\int_0^{\pi/2} e^{-aR\cdot\frac{2\theta}{\pi}} d\theta =$$

$$\frac{M\pi}{aR}\left(1-e^{-aR}\right) \xrightarrow{R\to\infty} 0$$

if $a>0$.





$\sin\theta \ge \frac{2}{\pi}\theta$, $0 \le \theta \le \frac{\pi}{2}$

10.2/8

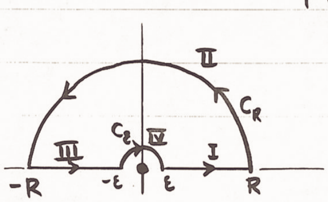
<u>Ex</u> $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,dx$, $a>0$ (Principal value)

Consider $\displaystyle\oint \frac{e^{iaz}}{z}\,dz = PV\int_{-R}^{R}\cdots + \int_{C_R}\cdots +$

$\displaystyle + \int_{C_\varepsilon} = 0$ (no poles inside)

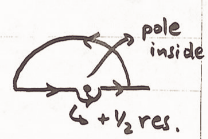
Since $\displaystyle\lim_{R\to\infty}\int_{C_R} = 0$ by Jordan lemma,

$$PV\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,dx = \pi i\ \text{Res}(z=0) = \pi i$$



i.e. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$

(what if we used:



<u>Note</u>: C_ε contributes $-\frac{1}{2}$ the residue at $z=0$

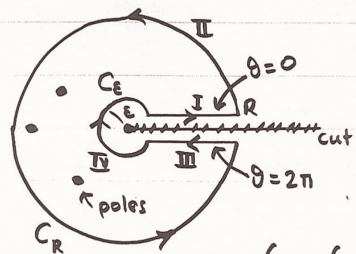
10.3/8

(6) $I = \int_0^\infty x^{a-1} f(x)\,dx \quad , \; 0<a<1$

$|f(z)| \to 0$ as $|z| \to \infty$; only poles ("meromorphic").

Consider $\oint z^{a-1} f(z)\,dz = 2\pi i \sum \text{Res}(\cdots)$ all poles

(if poles on + real axis, take principal values, get ½ residue from each side of branch cut). Can show:



(i) $C_I : \int_\varepsilon^R \cdots \longrightarrow I$ as $R \to \infty, \varepsilon \to 0$

(ii) on C_{III}, $z = re^{i2\pi}$, $z^{a-1} = r^{a-1} e^{i2\pi(a-1)}$

$\int_{III} \Rightarrow -e^{i2\pi(a-1)} I$

while $\int_{II}, \int_{IV} \to 0$ as $R \to \infty$, $\varepsilon \to 0$

10.4/8

so: $\oint_{III} \longrightarrow (1 - e^{i2\pi(a-1)}) I = 2\pi i \sum(\text{Res})$.

Ex. $\int_0^\infty \frac{x^{a-1}}{x+1}\,dx$: $\oint \frac{z^{a-1}}{z+1}\,dz =$

$= 2\pi i \cdot z^{a-1} \Big|_{z = e^{i\pi}} = 2\pi i e^{i\pi(a-1)}$

pole: $z = -1$

(I) $\int_\varepsilon^R \to I$

(III) $\int_R^\varepsilon \frac{(re^{i2\pi})^{a-1} \overbrace{e^{i2\pi}}^{1} dr}{r\underbrace{e^{i2\pi}}_{1} + 1} \Rightarrow -e^{i2\pi(a-1)} I$

on II : $\left| \int_0^{2\pi} \cdots \right| \le \int_0^{2\pi} \frac{|Re^{i\theta}|^{a-1} R\,d\theta}{|Re^{i\theta}+1|} \le \int_0^{2\pi} \frac{R^{a-1} R}{R-1}\,d\theta = \frac{2\pi R^a}{R-1} \xrightarrow{R\to\infty} 0 \quad (a<1)$

on IV : $\left| \int_{2\pi}^{0} \cdots \right| \le \int_0^{2\pi} \frac{|\varepsilon e^{i\theta}|^{a-1} \varepsilon\,d\theta}{|1+\varepsilon e^{i\theta}|} \le \frac{\varepsilon^a}{1-\varepsilon} \cdot 2\pi \xrightarrow{\varepsilon\to 0} 0 \quad (a>0)$

So: $(1 - e^{i2\pi(a-1)}) I = 2\pi i \cdot e^{i\pi(a-1)} \Rightarrow I = \frac{2\pi i e^{i\pi(a-1)}}{1 - e^{i2\pi(a-1)}}$

$\Rightarrow I = \frac{\pi}{\sin \pi(1-a)}$

10.5/8

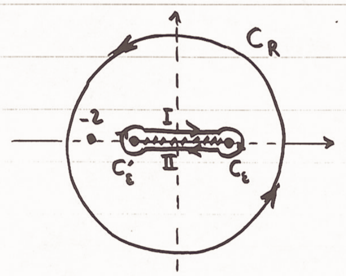
(7) Special problems

(a) $\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1-x^2}}$ integrate around cut

$(\,|f| \to 0,\ |z| \to \infty)$

Ex. $I = \int_{-1}^{1} \frac{dx}{(x+2)\sqrt{1-x^2}}$

Consider $\oint \frac{dz}{(z+2)\sqrt{1-z^2}} =$



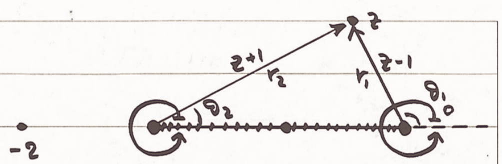
$= 2\pi i \cdot \text{Res}(-2) = 2\pi i \cdot \left.\frac{1}{\sqrt{1-z^2}}\right|_{-2}$

$= 2\pi i \cdot \frac{1}{\pm i\sqrt{3}} = \pm \frac{2\pi}{\sqrt{3}}$

<u>Problem</u>: we must define sq. root so we get (+) value for I!

As $R \to \infty$, $\varepsilon \to 0$, integrals around $C_\varepsilon, C_\varepsilon', C_R \to 0$.

10.6/8



Need to define $f(z) = (1-z^2)^{-1/2}$ so that

$f(-2) = +\frac{1}{i\sqrt{3}}$

Now $(1-z^2)^{1/2} = \pm i(z^2-1)^{1/2} = \pm i(r_1 r_2)^{1/2} e^{i\left(\frac{\vartheta_1+\vartheta_2}{2}\right)}$

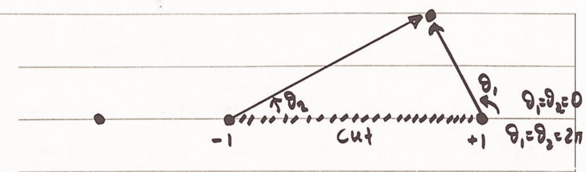
Choosing $0 \le \vartheta_1, \vartheta_2 < 2\pi$ gives:

$\left.(1-z^2)^{1/2}\right|_{-2} = \pm i\sqrt{3}\, e^{i\,2\pi/2} = \mp i\sqrt{3}$

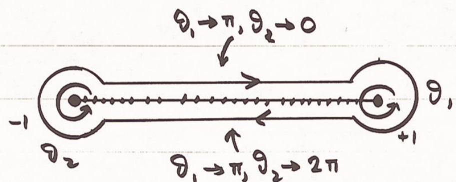
So, we will employ the definition

$(1-z^2)^{1/2} = -i\,(z^2-1)^{1/2}, \quad 0 \le \vartheta_1, \vartheta_2 < 2\pi$

10.7/8



The cut extends only between $(-1, +1)$: since both ϑ_1, ϑ_2 jump by 2π, $f(z)$ is continuous on real axis, $x > +1$!



Top: $f(z) = -i\left(|z-1||z+1|\right)^{1/2} e^{i\frac{\pi+0}{2}} = \sqrt{1-x^2}$ (+, real)

Bottom: $f(z) = -i\left(|z-1||z+1|\right)^{1/2} e^{i\frac{\pi+2\pi}{2}} = (-i)^2\sqrt{1-x^2} = -\sqrt{1-x^2}$

10.8/8

So: $\displaystyle\int_{C_\ell} \cdots \xrightarrow{\varepsilon \to 0} \int_{-1}^{+1} \frac{dx}{(x+2)\sqrt{1-x^2}} = I$

$\displaystyle\int_{C_u} \cdots \xrightarrow{\varepsilon \to 0} -\int_{+1}^{-1} \cdots = I$

Also

$\displaystyle\left|\int_{C_R} \cdots\right| \le \int_0^{2\pi} \frac{R\,d\vartheta}{(R-2)(R^2-1)^{1/2}} \xrightarrow{R\to\infty} 0$

$\displaystyle\left|\int_{C_\varepsilon} \cdots\right| \le \left|\int_{\pi}^{-\pi} \frac{\varepsilon\, d\vartheta}{(2-\varepsilon)\,\varepsilon^{1/2}\cdot(1-\varepsilon)^{1/2}}\right| \xrightarrow{\varepsilon\to 0} 0$; same for C_ε'.

So $\displaystyle 2I = \oint \cdots = 2\pi i \cdot \frac{1}{i\sqrt{3}} \Rightarrow I = \frac{\pi}{\sqrt{3}}$