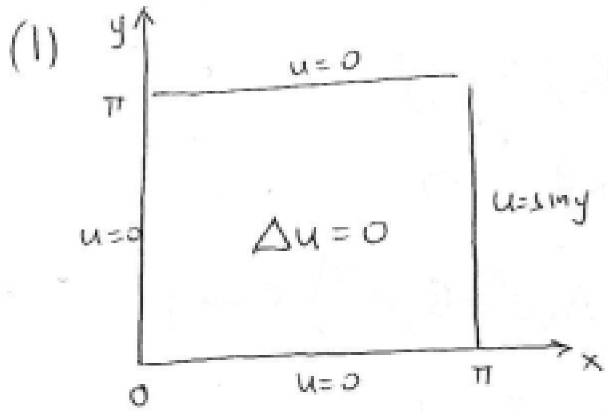
Solutions

(1)



Try sine series in y (since u vanishes at two points)

$$\frac{2}{\pi}\int_0^\pi \left(u_{xx}+u_{yy}=0\right)\sin ny\,dy \Rightarrow$$

$$\frac{d^2}{dx^2}\left(\frac{2}{\pi}\int_0^\pi u\sin ny\,dy\right) + \frac{2}{\pi}\int_0^\pi u_{yy}\sin ny\,dy = 0$$

$$\|$$

$$\frac{2}{\pi}\underbrace{\sin ny\, u_y\Big|_0^\pi}_{=0} - \frac{2}{\pi}\int u_y\,d(\sin ny)$$

$$= -\frac{2n}{\pi}\underbrace{u\cos ny\Big|_0^\pi}_{=0} - n^2\left(\frac{2}{\pi}\int_0^\pi u\sin ny\right)$$

$$\Rightarrow \frac{d^2u_n}{dx^2} - n^2 u_n = 0$$

$$\Rightarrow u_n(x) = A_n e^{nx} + B_n e^{-nx}$$

Then $$u(x,y) = \sum_{n=1}^{\infty}\left(A_n e^{nx} + B_n e^{-nx}\right)\sin ny$$

$$u(0,y) = 0 = \sum_{n=0}^{\infty} (A_n + B_n) \sin ny$$

$$\Rightarrow A_n + B_n = 0$$

$$u(\pi, y) = \sin y = \sum_{n=0}^{\infty} (A_n e^{\pi} + B_n e^{-\pi}) \sin ny$$

$$\Rightarrow A_1 e^{\pi} + B_1 e^{-\pi} = 1$$

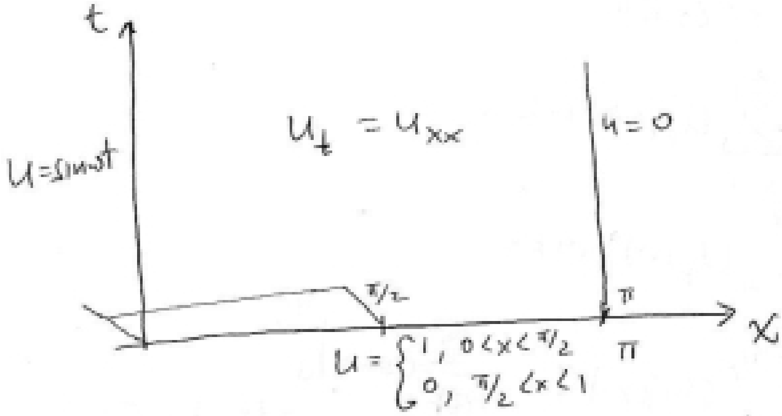
$$A_k e^{\pi} + B_k e^{-\pi} = 0, \quad k = 2, 3, \ldots$$

i.e. $\left.\begin{array}{l} A_1 + B_1 = 0 \\ A_1 e^{\pi} + B_1 e^{-\pi} = 1 \end{array}\right\}$ $\begin{array}{l} A_1 = -B_1 \Rightarrow B_1 = \frac{-2}{\sinh\pi} \\ A_1 = \frac{1}{e^{\pi} - e^{-\pi}} = \frac{2}{\sinh\pi} \end{array}$

$$\left.\begin{array}{l} A_k + B_k = 0 \\ A_k - B_k = 0 \end{array}\right\} \Rightarrow A_k = B_k = 0, \quad k = 2, 3, \ldots$$

$$\therefore \quad u(x,y) = \frac{\sinh x}{\sinh \pi} \sin y$$

(2)



$$\frac{2}{\pi}\int_0^\pi (u_t = u_{xx}) \sin kx\,dx \Rightarrow \frac{d}{dt}\left(\left(\frac{\pi}{2}\right)^{-1}\int_0^\pi u \sin kx\,dx\right) =$$

$$= \left(\frac{\pi}{2}\right)^{-1}\int_0^\pi u_{xx}\sin kx\,dx =$$

$$= \left(\frac{\pi}{2}\right)^{-1} \underbrace{u_x \sin kx\Big|_0^\pi}_{=0} - \left(\frac{\pi}{2}\right)^{-1} k\int_0^\pi u_x\cos kx\,dx$$

(everywhere. $\frac{\pi}{2} \to \frac{2}{\pi}$)

$$= -\left(\frac{\pi}{2}\right)^{-1} k\, u(x,t)\cos kx\Big|_0^{\pi \to =0} - k^2\left(\frac{\pi}{2}\right)^{-1}\int_0^\pi u\sin kx\,dx$$

$$= \left(\frac{\pi}{2}\right)^{-1} k\sin\omega t - k^2 u_k$$

Since $u(0,t)=\sin\omega t$ and $u_k = \left(\frac{\pi}{2}\right)^{-1}\int_0^\pi u\sin kx\,dx$

$$\Rightarrow \frac{du_k}{dt} = -k^2 u_k + \frac{2k}{\pi}\sin\omega t$$

Initial conditions

$$u(x,0) = \begin{cases} 1, & 0<x<\pi/2 \\ 0, & \pi/2<x<\pi \end{cases}$$

Now $\quad u(x,t) = \sum_{k=1}^{\infty}{}' \, u_k(t) \sin kx$

so $\quad u(x,0) = \sum_{k=1}^{\infty}{}' \, u_k(0) \sin kx$

$$u_k(0) = \frac{2}{\pi}\int_0^{\pi/2} \sin kx \, dx = -\frac{2}{\pi k}\cos kx \Big|_0^{\pi/2}$$

$$= \frac{2}{\pi k}\left(1-\cos\frac{k\pi}{2}\right) = \frac{2}{\pi k} - \frac{2}{\pi k}\begin{cases} 1, & k=0,4,8,\ldots \\ 0, & k=1,5,9,\ldots \\ -1, & k=2,6,10,\ldots \\ 0 & k=3,7,11,\ldots \end{cases}$$

i.e. $\quad u_{2k}(0) = \frac{2}{\pi k}\, u_{2l}(0) = \begin{cases} \cancel{}, & l = \text{even} \\ 0, & l = \text{odd} \end{cases}$

Best to leave in form

$$\boxed{u_k(0) = \frac{2}{\pi k}\left(1-\cos\frac{k\pi}{2}\right)}$$

Now: $\quad \frac{du_k}{dt} + k^2 u_k = \frac{2k}{\pi}\sin\omega t$

$\therefore \quad u_k(t) = A e^{-k^2 t} + B\sin\omega t + C\cos\omega t$

Substituting:

$$\frac{du_k}{dt} + k^2 u_k = -k^2\left(\cancel{Ae^{-k^2t}}\right) + k^2\left(\cancel{Ae^{-k^2t}}\right)$$

$$\|$$

$$\omega(B\cos\omega t - C\sin\omega t) + k^2(B\sin\omega t + C\cos\omega t) = \frac{2k}{\pi}\sin\omega t$$

$$\Rightarrow \quad \sin\omega t\left(\omega C + k^2 B\right) + (\omega B + k^2 C)\cos\omega t = \frac{2k}{\pi}\sin\omega t$$

$$\Rightarrow \quad \omega C + k^2 B = \frac{2k}{\pi}$$

$$\omega B + k^2 C = 0 \Rightarrow B = -\frac{k^2}{\omega}C$$

$$\Rightarrow \left(\omega - \frac{k^4}{\omega}\right)C = \frac{2k}{\pi}$$

$$\left.\begin{aligned} \Rightarrow \quad C &= \frac{2k\omega}{\pi(\omega^2 - k^4)} \\ B &= -\frac{2k^3}{\pi(\omega^2 - k^4)} \end{aligned}\right\}$$

i.e. $$u_k(t) = A_k e^{-k^2 t} + \frac{2k}{\pi(\omega^2 - k^4)}\left\{\omega\cos\omega t - k^2\sin\omega t\right\}$$

$$u_k(0) = \frac{2}{\pi k}\left(1 - \cos\frac{k\pi}{2}\right) = A_k + \frac{2k\omega}{\pi(\omega^2 - k^4)}$$

$$\Rightarrow \boxed{A_k = \frac{2}{\pi k}\left(1 - \cos\frac{k\pi}{2}\right) - \frac{2k\omega}{\pi(\omega^2 - k^4)}}$$

(3)

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\, d\theta$$

Mean value theorem: value at center = average of fnc. on boundary.

Here $u(x,y) = xy$ on x^2+y^2 ; convert to polar $u(r,\theta)\Big|_{r=1} = (r\cos\theta)(r\sin\theta)\Big|_{r=1} = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$

so $u(0,0) = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta\, d\theta = 0$

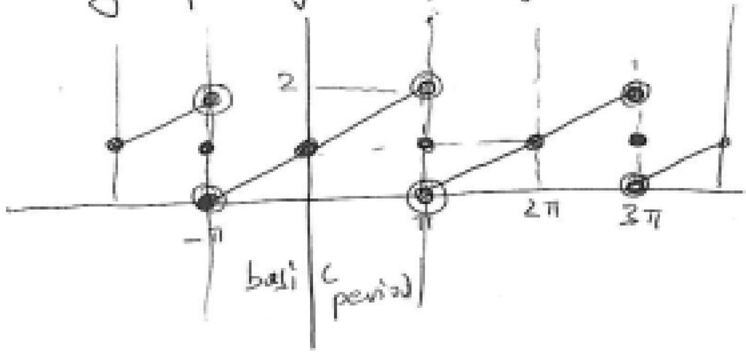
---

(4) The fourier series

$$f(x) \sim a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

describes a function on $-\pi < x < \pi$, with is 2π - periodic i.e. $f(x) = f(x+2\pi$

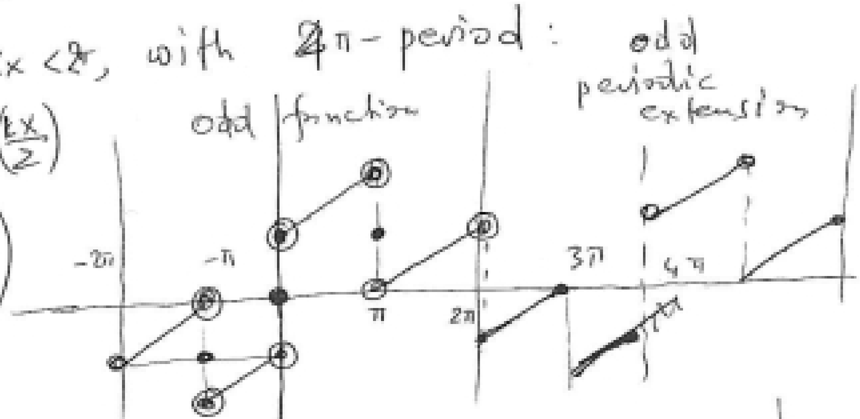
Now The graph of this fun. is therefore:



b) The sine series defines an odd function on $-2\pi < x < 2\pi$, with 4π-period:

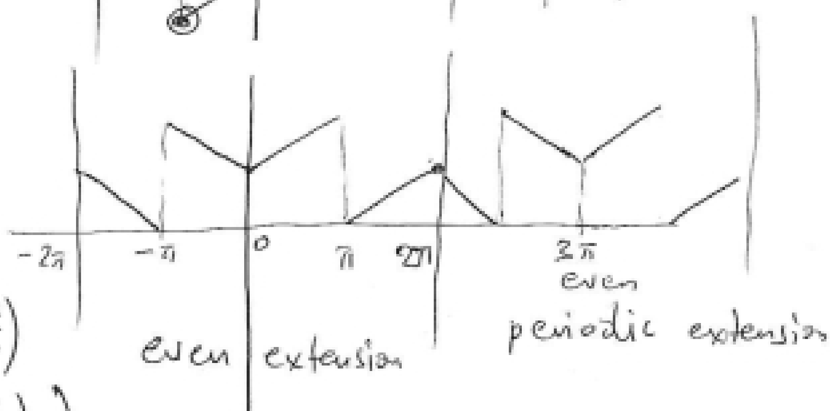
$f \sim \sum_{k=1}^{\infty} B_k \sin\left(\frac{kx}{2}\right)$

$\left(\sin\left(\frac{k\pi x}{2\pi}\right)\right)$



c)

Cosine series



$f \sim \sum_{k=0}^{\infty} A_k \cos\left(\frac{kx}{2}\right)$

$\left(\cos\left(\frac{k\pi x}{2\pi}\right)\right)$