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Extension of the gridless vortex method into the compressible flow regime*

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Abstract. This paper proposes an extension of the gridless vortex method into the compressible flow regime. The proposed method consists of tracking particles in the flow that carry vorticity, divergence, temperature and density. The particle velocity is given by the Helmholtz decomposition law, which is approximated using the trapezoid rule. The evolution equations for the particle vorticity, divergence, temperature and density are evaluated using finite differences or least squares approximations for all derivatives. The method is applied to an isentropic model problem and compared to solutions obtained using an Eulerian scheme. Difficulties with the least squares approximation and with boundary conditions are discussed.

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Contents

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1. Introduction

We propose a Lagrangian particle method to compute compressible fluid motion. The method consists of tracking particles together with their values of vorticity, divergence, temperature and density, and computing the particle velocity using the Helmholtz decomposition law. The Lagrangian quantities associated with the particles satisfy a set of evolution equations that reduces to a system of ordinary differential equations upon discretization. The potential advantages of such a method are that

(1) particles need only be placed in those regions of nonzero vorticity and divergence, thus leading to larger efficiency,
(2) regions of large gradients can easily be resolved by large particle densities and
(3) no grid-fitting to immersed boundaries is required.

The method is a natural extension of the incompressible vortex method to the compressible regime. Such extensions have been explored in the past, although few of those works resulted in truly gridless methods. The works of Sod [1] and Ogami and Cheer [2] require either a background grid fitted to boundaries or particles distributed everywhere. Mas-Gallic *et al* [3] used a semi-Lagrangian particle-in-cell method in which the velocity is computed on a fixed grid while the vorticity is carried by Lagrangian particles. Two truly gridless approaches were recently presented by Eldredge [4] and Eldredge *et al* [5], and by Strickland [6]. Both methods use the same idea, also explored by Quackenbush *et al* [7], of tracking particle vorticity and divergence, and using the Helmholtz decomposition law for the velocity. However, the implementation details in these approaches differ. Here, we follow the ideas presented in Strickland [6].

2. Governing equations

Let $\boldsymbol{u}(\boldsymbol{x},t)$, $\rho(\boldsymbol{x},t)$, $p(\boldsymbol{x},t)$, $e(\boldsymbol{x},t)$ and $T(\boldsymbol{x},t)$ be the fluid velocity, density, pressure, energy and temperature respectively. The equations governing viscous compressible flow are [8]

$$\begin{aligned}
\frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho(\nabla\cdot\boldsymbol{u}),\\
\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} &= -\frac{\nabla p}{\rho} + \frac{1}{3}\nu\nabla(\nabla\cdot\boldsymbol{u}) + \nu\Delta\boldsymbol{u},\\
\frac{\mathrm{D}e}{\mathrm{D}t} &= -\frac{p}{\rho}(\nabla\cdot\boldsymbol{u}) + \frac{1}{\rho}\nabla\cdot(k\nabla T) - \frac{2}{3}\nu(\nabla\cdot\boldsymbol{u})^2 + \nu(\nabla\boldsymbol{u} : (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)),\\
e &= C_vT, \qquad p = \rho RT
\end{aligned} \tag{2.1}$$

where $\nu = \mu/\rho$ and we have assumed that the first and second viscosity coefficients λ and μ are constant and satisfy $\lambda = -\frac{2}{3}\mu$ (Stokes hypothesis).

A discussion of the numerical method for the general nonisentropic, viscous case (2.1) can be found in [6]. In this paper we consider an isentropic model problem, for which

$$\frac{\nabla p}{\rho} = C_p\nabla T, \tag{2.2}$$

where $C_p = R + C_v$. This assumption is consistent with simultaneously neglecting conduction heat transfer and dissipation. Thus, for isentropic flow, (2.1) simplifies to inviscid equations for velocity and temperature. After nondimensionalizing using characteristic length, temperature and time scales L, T_∞ and $\sqrt{\gamma RT_\infty}/L$, where $\gamma = C_p/C_v$, the isentropic equations are

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\gamma-1}\nabla T, \tag{2.3}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t} = -(\gamma-1)T(\nabla\cdot\boldsymbol{u}). \tag{2.4}$$

The next section describes the numerical method applied to solve equations (2.3) and (2.4).

3. Particle method

The method is based on the fact that the fluid velocity is determined by the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ and the divergence $\delta = \nabla \cdot \boldsymbol{u}$ of the flow through the Helmholtz decomposition law:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \nabla \times \int \boldsymbol{\omega}(\boldsymbol{x}', t) K(\boldsymbol{x}, \boldsymbol{x}') \,\mathrm{d}V(\boldsymbol{x}') - \nabla \int \delta(\boldsymbol{x}', t) K(\boldsymbol{x}, \boldsymbol{x}') \,\mathrm{d}V(\boldsymbol{x}') + \nabla\phi,$$

$$K(\boldsymbol{x}, \boldsymbol{x}') = \begin{cases} \dfrac{1}{2\pi} \ln |\boldsymbol{x} - \boldsymbol{x}'| & \text{for 2D} \\ \dfrac{1}{4\pi} \dfrac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} & \text{for 3D,} \end{cases} \tag{3.1}$$

where ϕ vanishes for bodies moving in otherwise undisturbed fluid, and $K(\boldsymbol{x}, \boldsymbol{x}')$ is the Green function for the Laplace equation. The main idea is therefore to track particles in the fluid that carry the fluid divergence and vorticity, and recover the particle velocity using (3.1). It is also necessary to track the particle temperature (and density, in the nonisentropic case), since their divergence and vorticity depends on it, as shown by the following evolution equations:

$$\frac{\mathrm{D}\delta}{\mathrm{D}t} = -\frac{1}{\gamma - 1} \Delta T - \nabla \boldsymbol{u} : \nabla \boldsymbol{u}, \tag{3.2}$$

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - \boldsymbol{\omega}\delta, \tag{3.3}$$

where $\nabla \boldsymbol{u} : \nabla \boldsymbol{u} = \sum_i \sum_j \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$. These equations are obtained by taking the divergence and curl of equation (2.3) and using the identities $\nabla \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \boldsymbol{u} : \nabla \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\delta$ and $\nabla \times (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}\delta$.

In summary, the method consists of evolving the particle position using equation (3.1) and the associated values of divergence, vorticity and temperature using equations (3.2), (3.3) and (2.4). Note that the particle density satisfies the first equation in (2.1). This is implicitly satisfied in the isentropic case through the relation $\rho = T^{1/(\gamma-1)}$. The differential form of mass conservation therefore is satisfied exactly at the particles' position, although the integral form $\frac{\mathrm{d}}{\mathrm{d}t} \int_{W_t} \rho(\boldsymbol{x}, t) \,\mathrm{d}V(\boldsymbol{x}) = 0$ is only approximately satisfied.

The particle velocity is computed by approximating the integrals in (3.1) using the trapezoid rule. The rates of change (3.2), (3.3) and (2.4) are computed by approximating the derivatives on the right-hand side. The example presented below (with one spatial variable) uses finite difference approximations for these derivatives. In higher dimensions, we plan to use least squares approximations instead. The resulting system of ordinary differential equations for the particles' position, divergence, vorticity and temperature is solved using the fourth order Runge–Kutta method. The next section gives details of the method as applied to a model problem.

4. Model problem

We consider inviscid isentropic flow in two dimensions driven by a radially symmetric swirl velocity profile. The flow remains radially symmetric for all times, and the velocity and vorticity can be described in polar coordinates as functions of one spatial variable only, $\boldsymbol{u}(r, t) = u(r, t)\boldsymbol{e}_r + v(r, t)\boldsymbol{e}_\theta$, and $\boldsymbol{\omega}(r, t) = \omega(r, t)\boldsymbol{e}_z$. The initial conditions are a swirling flow with constant temperature, given by

$$u(r, 0) = 0, \qquad v(r, 0) = r^2 \mathrm{e}^{-b(r - r_c)^2}, \qquad T(r, 0) = 1. \tag{4.1}$$

The initial vorticity and divergence distribution can be deduced to be equal to

$$\begin{aligned} \delta(r,0) &= \frac{1}{r}\frac{\partial}{\partial r}[ru(r,0)] = 0, \\ \omega(r,0) &= \frac{1}{r}\frac{\partial}{\partial r}[rv(r,0)] = \left[\frac{3}{r} - 2b(r-r_c)\right]v(r,0). \end{aligned} \tag{4.2}$$

To compute the flow with the proposed method, we track particles with position $r_j(t)$, divergence $\delta_j(t)$, vorticity $\omega_j(t)$ and temperature $T_j(t)$. The particles are initially uniformly positioned on an interval $[0, r_{max}]$, at $r_j(0) = j\, r_{max}/n$, $j = 0, \ldots n$. The values of $\delta_j(0)$, $\omega_j(0)$ and $T_j(0)$ are given by equations (4.1) and (4.2).

The Helmholtz decomposition law (3.1) simplifies in this case to

$$\begin{aligned} u_j(t) &= \frac{1}{2\pi r_j}\int_0^\infty \int_0^{2\pi} \delta(r)\frac{r_j^2 - rr_j\cos\theta}{r_j^2 + r^2 - 2rr_j\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{r_j}\int_0^{r_j} r\delta(r)\,\mathrm{d}r, \\ v_j(t) &= \frac{1}{2\pi r_j}\int_0^\infty \int_0^{2\pi} \omega(r)\frac{r_j^2 - rr_j\cos\theta}{r_j^2 + r^2 - 2rr_j\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{r_j}\int_0^{r_j} r\omega(r)\,\mathrm{d}r. \end{aligned} \tag{4.3}$$

The evolution of $r_j, \delta_j, \omega_j, T_j$ is given by

$$\begin{aligned} \frac{\mathrm{d}r_j}{\mathrm{d}t} &= u_j, \\ \frac{\mathrm{d}\delta_j}{\mathrm{d}t} &= -\frac{1}{\gamma - 1}\Delta T - \nabla\boldsymbol{u} : \nabla\boldsymbol{u}, \\ \frac{\mathrm{d}\omega_j}{\mathrm{d}t} &= -\omega_j\delta_j, \\ \frac{\mathrm{d}T_j}{\mathrm{d}t} &= -(\gamma-1)T_j\delta_j. \end{aligned} \tag{4.4}$$

The system (4.4) of ODEs is integrated in time using the Runge–Kutta method. At each stage of the Runge–Kutta method we first approximate the current values of u_j and v_j, given by (4.3), using the trapezoid rule. Then, we compute the current values of $\mathrm{d}\delta/\mathrm{d}t, \mathrm{d}\omega/\mathrm{d}t$ and $\mathrm{d}T/\mathrm{d}t$, given by (4.4), using second order finite difference approximations for the derivatives on the right-hand side. The method is second order accurate in space and stable as long as $\Delta t \le C\Delta r$ where C is a constant. It is interesting to note that the scheme is stable if Euler's method is used instead of Runge–Kutta for the time integration. This is attributed to the fact that in the Lagrangian reference frame, the advection terms, for which Euler's method is typically unstable, are removed. However, the timesteps required for stability using Euler's method are very small, $\Delta t \le C(\Delta r)^2$. This can probably be deduced from the relation between the real and imaginary components of the eigenvalues of the linearized system, but we have not yet performed that analysis.

For comparison, we also solve the governing equations in an Eulerian framework, on a fixed uniform mesh. Numerical stability is obtained by first diagonalizing the system and then applying a first order generalized upwind scheme. Figures 1(a)–(e) plot the computed vorticity, divergence, temperature and velocity components at times = 0:0.4:5.2, using both the particle method (solid curve) and the Eulerian scheme (dashed curve). The dashed curve is not visible since it falls on top of the solid curve. This confirms that the particle method gives the correct solution, by comparison with a method that is well understood. Note that for resolution the Eulerian scheme requires much larger values of n (see caption) than the particle method, since it is only of first order. However, the goal here is not to compare the efficiency of the methods, but to verify the correctness of the solution of the particle method.

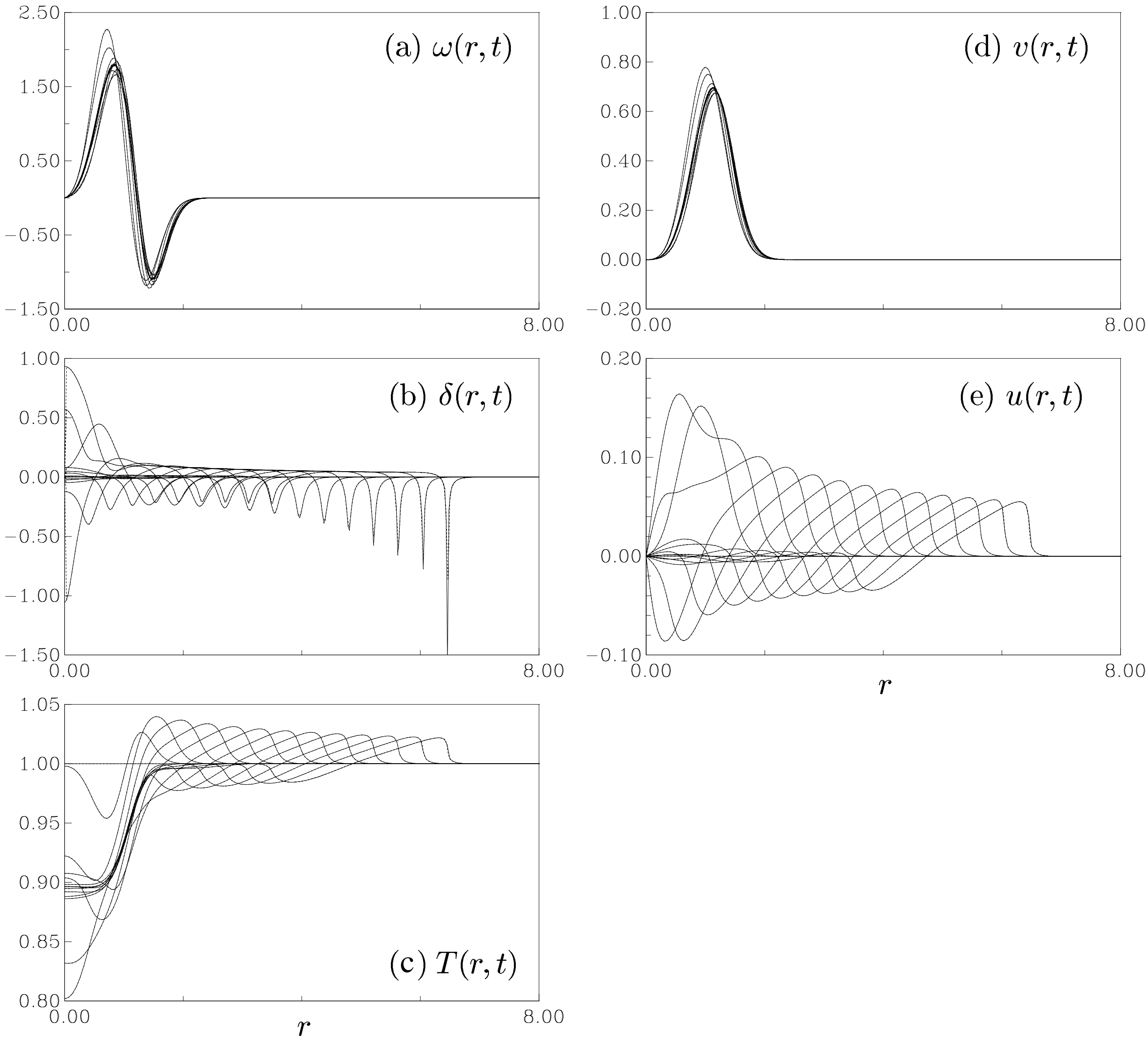

Figure 1. Solution at $t = 0{:}0.4{:}5.2$, computed with the particle method (solid curve) and with an Eulerian (generalized upwind) scheme (dashed curve). The particle method uses $n = 4000$, $\Delta t = 0.001$. The Eulerian scheme uses $n = 50\,000$, $\Delta t = 0.000\,025$. (a) $\omega(r,t)$; (b) $\delta(r,t)$; (c) $T(r,t)$; (d) $v(r,t)$; (e) $u(r,t)$.

A couple of comments about figure 1 are noteworthy. The velocity is nondimensionalized by the speed of sound $\sqrt{\gamma R T_\infty}$. Figures 1(d) and (e) indicate that the maximum velocity lies approximately in the 0.7–0.8 Mach number range, thus compressibility effects are rather large. Because of the compressibility, both the divergence and temperature develop waves that quickly travel outwards. The divergence develops a large negative peak, indicating the formation of a shock. The vorticity however barely changes, as it would in the incompressible case. Thus, the vorticity is relatively insensitive to compressibility effects, at least in this isentropic case. This is consistent with a statement by Lighthill [9] to this effect.

The radial speed of the peripheral divergence and temperature waves is expected to equal the radial fluid velocity plus the speed of sound. At the last time step shown, the radial fluid velocity in figure 1(e) approximately equals 0.06, which would give a wave speed of 1.06. This agrees with figures 1(b) and (c). The difference in the position of the divergence and temperature peaks per unit time is slightly larger than unity.

5. Concluding remarks

The previous section applied the proposed particle method to a model problem, and showed that the method gives an accurate solution. However, many questions regarding the method remain to be addressed. We conclude this paper by summarizing the unresolved issues.

The derivatives in the model problem presented above, a problem in one spatial variable, were computed using finite differences. This is impractical in higher dimensions. One alternative is the deterministic treatment described in Eldredge *et al* [10], in which the derivatives are computed as integral quantities over the whole domain. We intend to pursue the use of local least squares approximations for all derivatives. Preliminary studies have shown that this approach works well as long as there are no physical boundaries in the problem. The question of how to tackle derivatives near boundaries using least squares remains open. A second question regards boundary conditions. It is not yet clear what the correct boundary conditions are for a vorticity-divergence formulation as proposed here. Strickland *et al* [11] and Wolfe and Strickland [12] derived a condition on the vorticity gradient at the boundary that works well in conjunction with the diffusion velocity concept. Alternative boundary conditions on the vorticity are discussed in Cottet and Koumoutsakos [13], but a practical condition on the divergence remains to be found. Finally, as can be seen from the results presented here, the divergence expands rapidly. As a result it is costly, already after short times, to track all the region containing nonzero divergence. It may be possible to truncate or approximate this region without greatly affecting the local fluid dynamics. All of these issues remain to be explored.

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References

[1] Sod G A 1991 *Appl. Numer. Math.* **8** 257–73
[2] Ogami Y and Cheer A Y 1993 *AIAA J.* **31** 1155–7
[3] Mas-Gallic S, Louaked M and Pironneau O 1993 *Vortex Flows and Related Numerical Methods* ed J T Beale *et al* (London: Kluwer) pp 373–87
[4] Eldredge J D 2001 *PhD Thesis* California Institute of Technology
[5] Eldredge J D, Colonius T and Leonard A 2002 *J. Comput. Phys.* **179** 371–99
[6] Strickland J H 2001 *Sandia National Laboratory Report* SAND2001-0527
[7] Quackenbush T R, Boschitsch A H, Winckelmans G S and Leonard A 1996 *AIAA Paper* 96-0815
[8] Curry I G 1974 *Fundamental Mechanics of Fluids* (New York: McGraw-Hill)
[9] Lighthill G 1978 *Waves in Fluids* **See endnote. 1**
[10] Eldredge J D, Leonard A and Colonius T 2002 *J. Comput. Phys.* **180** 686–709
[11] Strickland J H, Kempka S N and Wolfe W P 1996 *ESAIM Proc.* **1** 135–51
[12] Wolfe P W and Strickland J H 1999 *ESAIM Proc.* **7** 440–54
[13] Cottet G-H and Koumoutsakos P S 2000 *Vortex Methods: Theory and Practice* (Cambridge: Cambridge University Press)

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