MATH 163 — REVIEW EXAM 2: SOLUTIONS

Below are the final answers to most problems in the review. Most of them show all the detail that you need to show in your solutions.

0. (a) \((a/b) \ln |(x-b)/x| + C\)  
(b) \(x + \ln |x/(x-2)| - 2/x + C\)

1. (b) \(T_4 = ((1 + e)/2 + \sqrt{e} + e^{1/4}(1 + \sqrt{e}))/4 \approx 1.7272\) (Exact answer \(I = e \approx 1.7183\), Error \(|I - T_4| \approx 0.0089\))

(c) Overestimate (since \(e^x\) concave up)

(d) First, need to find upper bound for \(|f''(c)| = |e^c|\) for some \(c \in (0, 1)\). Since \(e^c\) is increasing, the largest \(|f''(c)|\) could be is at the right endpoint, \(c = 1\). So \(|f''(c)| < e^1 = 0.014\).

If \(n = 4\), it follows from this bound and the formula that \(|I - T_4| \leq e/(12 \cdot 16) \approx 0.0142\). The actual error is indeed smaller than this upper bound.

(e) If \(n = 8\), an upper bound for the error is \(|I - T_8| \leq e/(12 \cdot 64) \approx 0.0035\).

(f) \(|I - T_n| \leq 0.0001\) if \(|I - T_n| \leq e/(12 \cdot n^2) < 10^{-4}\)

which holds if \(n > \sqrt{10^4} \cdot e/12 \approx 100/\sqrt{4} = 50\). (In exact arithmetic, the value of \(n\) needed would be 48. but you do need to be able to estimate numbers)

2. If \(p = 1\) then the integral equals \(\lim_{R \to \infty} (\ln R - \ln a) = \infty\) so it diverges.

If \(p \neq 1\), the integral equals \(\frac{1}{p-1} \lim_{R \to \infty} (R^{-p+1} - a^{-p+1})\). This limit is infinite if \(p < 1\) since exponent \(1 - p > 0\), and finite if \(p > 1\) \((p - 1 > 0)\) since

\[\lim_{R \to \infty} \frac{1}{R^{p-1}} = 0\]

Summary: convergence if \(p > 1\), divergence if \(p \leq 1\).

3. (a) Diverges since \(\lim_{x \to \infty} f(x) \neq 0\).

(b) \(\int_0^1 \frac{x-1}{\sqrt{x}} \, dx = \lim_{a \to 0^+} \int_a^1 1/\sqrt{x} - 1/\sqrt{x} \, dx = \lim_{a \to 0^+} \left[ \frac{2}{3} x^{3/2} - 2x^{1/2} \right]_a^1 = \lim_{a \to 0^+} \left[ \left( \frac{2}{3} - 2 \right) - \left( \frac{2}{3} a^{3/2} - 2a^{1/2} \right) \right] = -\frac{4}{3}\)

(c) First find antiderivative \(\int \ln x \, dx = x \ln x - x + C\) using integration by parts. Then

\(\int_0^3 \ln(x) \, dx = \lim_{a \to 0^+} \int_a^3 \ln(x) \, dx = \lim_{a \to 0^+} [x \ln x - x]_a^3 = \lim_{a \to 0^+} (3 \ln 3 - 3 - a \ln a + a) = 3 \ln 3 - 3 - \lim_{a \to 0^+} \frac{\ln a}{1/a} = 3 \ln 3 - 3 - \lim_{a \to 0^+} \frac{1/a}{-1/a^2} = 3 \ln 3 - 3 + \lim_{a \to 0^+} a = 3 \ln 3 - 3\)

we used L'Hôpital’s rule to evaluate limit. (You do need to show all this work here.)

(d) \(\int_0^\infty \frac{x^2}{9 + x^6} \, dx = \frac{1}{3} \int_0^\infty \frac{du}{9 + u^2} = \frac{1}{2} \int_0^\infty \frac{du}{1 + (u/3)^2} = \lim_{R \to \infty} \frac{1}{9} \left[ \tan^{-1} \left( u/3 \right) \right]_0^R = \lim_{R \to \infty} \frac{1}{9} \left( \tan^{-1}(R/3) - \tan^{-1} 0 \right) = \frac{\pi}{18}\), where \(u = x^3\)

(e) Converges to \(2 \lim_{t \to 3^+} (\sqrt{3} - \sqrt{t-3}) = 2\sqrt{3}\)

(f) Converges to \(\frac{1}{a} \lim_{R \to \infty} (e^{-aR} - 1) = 1/a\)

(g) Diverges since \(\lim_{t \to 1^-} (\ln |\frac{t+1}{1-t}| - \ln 1) = \infty\)
(h) Diverges since \( \lim_{y \to -\infty} f(y) \neq 0. \)

\[ (i) \int_3^5 \frac{dx}{4x^2-9} = \int_3^5 \frac{dx}{(2x-3)(2x+3)} = \frac{1}{6} \int_3^5 \frac{1}{2x-3} - \frac{1}{2x+3} \, dx \]
\[ = \frac{1}{12} [\ln |2x-3| - \ln |2x+3|]_3^5 = \frac{1}{12} [\ln (\frac{7}{13}) - \ln (\frac{3}{9})] = \frac{1}{12} \ln (\frac{21}{13}) \]

4. Hint: rewrite as \( \overline{v} = a \int_0^\infty v^3 e^{-bv^2} \, dv \). Use the substitution \( u = v^2 \), use integration by parts, take limit and find \( \overline{v} = a/(2b^3) \). Substitute full expressions for \( a, b \) back in and get result.

5. Make sure to rewrite as a limit, and use the fact that the antiderivative of \( f' \) is \( f \).

6. For simplicity I will write \( F = a/r^2 \), where \( a = q/(4\pi\varepsilon_0) \). Now, let \( r_o \) be the distance between the charges \( P \) and \( q \). The potential \( V \) is

\[ \int_{r_o}^\infty F(r) \, dr = \lim_{R \to \infty} a \int_{r_o}^R \frac{dr}{r^2} = \lim_{R \to \infty} a [-1/r]_{r_o}^R = \lim_{R \to \infty} a(-1/R + 1/r_o) = a/r_0 = q/(4\pi\varepsilon_0) \]

7. Let \( y = y_1 \). Compute \( y' \) and \( y'' \) and substitute into equation to find that

\[ e^{rt} (r + 1)^2 \sin t + 2(r + 1) \cos t ) = 0 \]

\( y_1 \) is a solution if this equation is satisfied for all \( t \). That is the case if \( r = -1 \). So for this value of \( r \), \( y_1(t) \) is a solution. Same holds for \( y_2 \).

9. §9.3, #16: \( P(t) = (t^{3/2}/3 + \sqrt{2} - 1/3)^2 \)

10. (b) \[ \int \frac{dP}{P(2-P)} = \int dt \quad \Rightarrow \quad \frac{1}{2} \ln \left| \frac{P}{2-P} \right| = t + c \]

Since \( 0 < P(0) < 2 \) it follows from (a) that \( 0 < P(t) < 2 \) for all times, so \( |P| = P \) and \( |2-P| = 2 - P \). So (after multiplying by 2 and exponentiating):

\[ \frac{P}{2-P} = Ce^{2t} \quad \text{where} \quad C = e^{2c} > 0 \]

From initial condition it follows that \( C = 1/3 \). Solve for \( P \) to get \( P(t) = \frac{2e^{2t}}{3 + e^{2t}} \).

(c) \( \lim_{t \to \infty} P(t) = 2 \), \( \lim_{t \to -\infty} P(t) = 0 \).

(d) Eulers method: \( t_0 = 0, t_1 = 1/2, t_2 = 1 \). \( P_0 = 1/2, P_1 = 7/8, P_2 = 175/128 \approx 1.367. \)
\( P_2 \) approximates \( P(1) \) which equals \( 2e^2/(3 + e^2) \approx 1.422 \), based on the work in (b). So Euler’s method gives a little bit of an underestimate, which is expected based on direction field.

12. (b) \( P = 0 \)
(c) \( P(t) = M \)
(d) \( P(t) = M - (M - P_0)e^{-kt} \)
(e) \( kM \) is the rate of increase of performance when \( P = 0 \). So \( k \) measures rate of increase of performance when \( P = 0 \), per maximum possible.