TOPICS

20. Absolute and conditional convergence

Reading: §12.6, first half

○ **Absolute convergence.** Theorem: If \( \sum_{n=1}^{\infty} |a_n| \), then \( \sum_{n=1}^{\infty} a_n \) converges.

Definition: If \( \sum_{n=1}^{\infty} |a_n| \) converges, then we say \( \sum_{n=1}^{\infty} a_n \) converges **absolutely**.

○ **Conditional convergence.** If \( \sum_{n=1}^{\infty} a_n \) converges but \( \sum_{n=1}^{\infty} |a_n| \) does not converge, then we say the series converges **conditionally**.

○ **To check convergence of a series with negative terms:**

After checking for divergence using Divergence Test

- *First check for absolute convergence*, that is, check convergence of the series with positive terms. For this you use any of the other tests we have seen, including the Ratio and Root test below.

- *Only if a series does not converge absolutely* and it happens to be alternating, then apply the alternating series test, to see if it converges conditionally.

- *If the alternating series test fails* (and you already checked that it does not converge absolutely), then you cannot conclude anything about conditional convergence! All this is summarized in the flowchart posted on the web.

○ **To evaluate series:**

- If a series is alternating and satisfies the alternating series test, you can approximate its value using the Alternating series estimation test (whether or not it converges absolutely).

- If a series is positive and satisfies the Integral test, you can approximate its value using the Integral Estimation Test.

21. Ratio and Root tests

Reading: §12.6, second half

○ **Ratio test.** Consider any series \( \sum_{n=1}^{\infty} a_n \). Let \( r = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \).

  (i) If \( r < 1 \) then the series converges (absolutely). Reason: its tail \( \sum_{n=N}^{\infty} a_n \) looks like the geometric series \( a_N \sum_{n=N}^{\infty} r^n \) for some sufficiently large \( N \).

  (ii) If \( r > 1 \) then the series diverges.

  (iii) If \( r = 1 \) then the test is inconclusive.

○ **Root test.** Consider any series \( \sum_{n=1}^{\infty} a_n \). Let \( L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \).

  (i) If \( L < 1 \) then the series converges (absolutely).

  (ii) If \( L > 1 \) then the series diverges.

  (iii) If \( L = 1 \) then the test is inconclusive.
22. Power series

Reading: §12.8

○ Definition.
A power series about \( x = 0 \) is an expression of the form \( \sum_{n=0}^{\infty} c_n x^n \).

A power series about \( x = a \) is an expression of the form \( \sum_{n=0}^{\infty} c_n (x - a)^n \).

○ Radius and interval of convergence. If a power series about \( x = a \) converges at \( |x - a| = t \), then it converges at all \( |x - a| = s \) with \( 0 \leq s < t \). As a result, a power series about \( 0 \) converges at either
- at a single point \( x = a \), or
- in an interval of radius \( R \) about \( x = a \), that is, \((a - R, a + R)\) plus possibly one or both endpoints. The series will converge absolutely in the interior of the interval and possibly only conditionally at the endpoints. Or
- all values of \( x \) (an interval of infinite radius).
These are the only possibilities.

○ Finding the radius and interval of convergence. In general, we will use the Ratio Test to find the radius of convergence. We may have to use the alternating series test to check for convergence at the endpoints.

23. Power series representation of functions

Reading: §12.9

○ An example. We already know the power series representation of one function

\[ f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1, \]

and that the series converges to this function for \( |x| < 1 \). Starting with this example we can obtain representations of new functions using substitution, addition, multiplication, differentiation, integration. (See below.)

○ Important application. If \( f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \) converges on an open interval \( I \), then we can approximate \( f \) by a polynomial! and, as we will see, we will be able to estimate the error in this approximation. (This is actually how the calculator evaluates many transcendental functions. Polynomial approximations are also used to simplify differential equations, approximate derivatives numerically, estimate integrals, estimate physical quantities. Many uses.)

○ Obtaining new power series from old by addition and scalar multiplication. Suppose two series about the same point \( \sum c_n (x - a)^n \) and \( \sum d_n (x - a)^n \) converge absolutely in an interval \( I \). Then

(i) The sum of the series converges absolutely in \( I \) and equals the series of the sum,
\[ \sum c_n (x - a)^n + \sum d_n (x - a)^n = \sum (c_n + d_n)(x - a)^n \]

(ii) The scalar product \( \lambda \sum c_n (x - a)^n \) converges absolutely in \( I \) and equals the series of the scalar products, \( \sum \lambda c_n (x - a)^n \).

(iii) The product \( (x - a)^k \lambda \sum c_n (x - a)^n \) converges absolutely in \( I \) and equals the series of the products, \( \sum c_n (x - a)^{(n+k)} \).
Obtaining new power series from old by differentiation and integration. Let \( f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \) with radius of convergence \( R \). In the interval \( I = (c - R, c + R) \):

(i) the derivative of the series equals the series of the derivatives,

\[
f'(x) = \sum_{n=0}^{\infty} na_n (x - c)^{n-1}.
\]

(ii) the integral of the series equals the series of the integrals,

\[
\int f(x) \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1} + C.
\]

The radius of convergence of these new series is also \( R \). They may or may not converge at the endpoints of \( I \).

PROBLEMS

DAY 30: Absolute convergence. Ratio Test, Root Test.

§12.6: # 1, 2, 3, 4, 5, 6, 7, 8, 10, 15

2. Determine whether the following series converge absolutely, conditionally, or diverge.

(a) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln^3 n} \)

(b) \( \sum_{n=1}^{\infty} (-1)^n n 2^{-n^2} \)

(c) \( \sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n} \)

DAY 31: Power series.

1. Which of the following are power series?

(a) \( \sum_{n=0}^{\infty} (3x)^n \)

(b) \( \sum_{n=0}^{\infty} \sqrt{n} x^n \)

(c) \( \sum_{n=3}^{\infty} (x + 2)^{2n} \)

2. Determine the radius and the interval of convergence of the following power series.

(a) \( \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}} \)

(b) \( \sum_{n=0}^{\infty} (-1)^n 4^n x^n \)

(c) \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \)

(d) \( \sum_{n=0}^{\infty} \frac{(-1)^n (x - 3)^n}{2n + 1} \)

(e) \( \sum_{n=1}^{\infty} \ln(n) (x + 2)^n \)

(f) \( \sum_{n=0}^{\infty} n! (2x - 1)^n \)

3. The power series \( \sum_{n=1}^{\infty} \frac{4^n (x + 3)^{2n}}{n^2} \)

comprises only even powers of the form \( (x - c)^{2n} \). What is \( c \)? Make the substitution \( t = (x - c)^2 \) and find the interval of convergence of the simpler series in \( t \). Then use your result to find the interval of convergence of the original series.

4. §12.8: 29

5. §12.8: 30
DAY 32: Representing functions as series. Obtaining series representations of functions using known representations.

1. Starting with the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $-1 < x < 1$, find power series representations about $x = 0$ of the following functions using substitution, addition, and multiplication. Also state the radius and interval of convergence.

   (a) $\frac{1}{1+x}$  
   (b) $\frac{x^2}{1+x^2}$  
   (c) $\frac{1+x}{1-x}$  
   (d) $\frac{1}{1+x^4}$  
   (e) $\frac{2}{3-x}$

2. Starting with the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $-1 < x < 1$, find power series representations about $x = 0$ of the following functions using integration or differentiation, in addition to substitution, addition, multiplication. Also state the radius and interval of convergence.

   (a) $\frac{1}{(1-x)^3}$  
   (b) $\frac{1}{(1+2x)^2}$  
   (c) $\ln(1+x^2)$  
   (d) $\int_0^x \ln(1+t^2) \, dt$  
   (e) $\tan^{-1}(x^2)$  
   (f) $\ln(5-x)$