MATH 163 —HOMEWORK # 9 : ANSWERS to some problems

Note: in most cases we DO NOT show all the work necessary to obtain the answer. You need to show all work.

Solutions Day 24:

1. (a) $5/3 = 1.6$
   (b) 3
   (c) 2
   (d) 4/15
   (e) diverges
   (f) $-1$
   (i) $3\pi/4 - \tan^{-1} 2$

Solutions Day 25:

1. (a) none ($\lim_{n \to \infty} a_n = 0$)
   (b) diverges ($\lim_{n \to \infty} a_n = \infty \neq 0$)
   (c) none ($\lim_{n \to \infty} a_n = 0$)
   (d) diverges ($\lim_{n \to \infty} a_n = 1 \neq 0$)
   (e) none ($\lim_{n \to \infty} a_n = 0$)
   (f) diverges ($\lim_{n \to \infty} a_n = 2/3 \neq 0$)
   (g) diverges ($\lim_{n \to \infty} a_n = \pi/2 \neq 0$)
   (h) diverges ($\lim_{n \to \infty} a_n = 1 \neq 0$)

2. (a) may  (b) must  (c) may  (d) may  (e) must

3. (a) $3 + 2 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 9$ (basic property iv)
   (c) $\frac{1}{100} + \frac{1}{9} = 0.12\bar{2}$ (basic properties iii and iv)

4. (a) The tail of this series is the same as the tail of the harmonic series.
   (b) Converges (telescoping, value = 1). Basic property (iii) does not say anything about the sum or difference of two diverging series.
   (c) Diverges since it is a difference of a diverging sequence and a converging sequence. There is nothing to cancel the divergence of first part.

5. (a) $2(0.1111\ldots) = 2 \sum_{n=1}^{\infty} \frac{1}{10^n} = 2/9$

7. (a) $r = -1 + \sqrt{3}$
   (b) $c = \frac{\sqrt{3} - 1}{2} \approx 0.366$

Solutions Day 26:

2. (c) Let $f(x) = 1/\sqrt{x}$. Note
   (i) $f(k) = a_k \geq 0$, for $k \geq 1$ (all you need is that $a_k \geq 0$ for $k$ sufficiently large, that is, bigger than some $N$). Also
   (ii) $f'(x) = -1/(5\sqrt{x^4}) \leq 0$ for $x > 0$ (again, all you really need is that $f' \leq 0$ for $x > N$ some $N$).
So the two conditions for the integral test apply. Since
\[ \int_{4}^{\infty} f(x) \, dx = \infty \]
the series \( \sum_{k=4}^{\infty} a_k \) diverges. (The series diverges no matter what the starting value is since the sum of the tail diverges.)

(d) Let \( f(x) = \frac{e^x}{(1 + e^x)^2} \). Note

(i) \( f(n) = a_n \geq 0 \) for any \( n \), and

(ii) \( f'(x) = e^x(1 - e^x)/(1 + e^x)^3 \leq 0 \) for \( x \geq 0 \).

So the two conditions for the integral test apply. Since
\[ \int_{0}^{\infty} f(x) \, dx = \int_{2}^{\infty} \frac{du}{u^2} = \lim_{t \to \infty} \int_{2}^{t} -\frac{1}{u} \, du = 1/2 \]
(Again, as long as \( f \) is continuous on the interval of integration, it doesn’t matter what the left endpoint is. I chose one that was convenient.) Since this integral is finite, the series \( \sum_{n=1}^{\infty} a_n \) converges.

3. 10: converges since series equals sum of two converging series with \( p = 1.2, 1.4 > 1 \).

14: series equals \( \sum_{n=0}^{\infty} \frac{1}{n^{1.3n}} \) which diverges by integral test (show all work)

4. First, note integral test applies: \( a_n \geq 0 \) and \( f(x) = 1/x^6 \) is decreasing. The first \( N \) for which \( \int_{N}^{\infty} f(x) \, dx = 1/(5N^5) \leq 10^{-6} \) is \( N = 12 \). Thus
\[ \sum_{n=1}^{\infty} \frac{1}{n^b} = \sum_{n=1}^{12} \frac{1}{n^b} + error = 1.017342 + error \]
where \( 0 \leq error \leq 10^{-6} \).

5. First, note integral test applies: \( a_n \geq 0 \) and \( f(x) = 1/x^2 \) is decreasing. The first \( N \) for which \( \int_{N}^{\infty} f(x) \, dx = 1/N \leq 0.05 \) is \( N = 20 \). Thus
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{20} \frac{1}{n^2} + error = 1.5962 + error \]
where \( 0 \leq error \leq 0.05 \).