TOPICS

16. Series $\sum_{n=1}^{\infty} a_n$ (ctd)

Reading: §12.2

- **Partial sums: definition.** Suppose $\{a_n\}_{n=1}^{\infty}$ is an infinite sequence. The $N$th partial sum is the finite sum of the first $N$ terms in the sequence:
  $$S_N = \sum_{n=1}^{N} a_n$$
  The set of partial sums $\{S_N\}_{N=1}^{\infty}$ forms an infinite sequence.

- **Series: definition.** The series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$ is an infinite sum defined to be the limit of the partial sums
  $$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N$$
  If the limit is finite and equal to $L$ we say the series converges to $L$. Otherwise the series diverges.

- **Geometric Series.** Derive the formula
  $$\sum_{n=0}^{\infty} r^n = \begin{cases} 1 & \text{if } |r| < 1 \\ \text{diverges if } |r| \geq 1 \end{cases}$$

- **Telescoping Series.** Recognize and evaluate telescoping series.

- **Basic properties.**
  (i) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$ (Divergence test)
  (ii) $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for any $N$ (convergence/divergence depends on behaviour of tail end of series only)

- **More basic properties.**
  (iii) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
  (iv) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (ca_n) = c \sum_{n=1}^{\infty} a_n$, where $c$ is a constant
  (v) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} (ca_n)$ diverges, provided $c \neq 0$.

17. Integral test

Reading: §12.3

- The integral test and the comparison tests (next week) only apply to series $\sum_{n=n_0}^{\infty} a_n$ with nonnegative terms $a_n \geq 0$.

- **Series with nonnegative terms** $a_n \geq 0$. For these series the sequence of partial sums is increasing, $S_{N+1} \geq S_N$, all $N$. An increasing sequence of partial sums converges if and only if it is bounded, $S_N \leq M$, all $N$, some $M$.

- **Integral Test.** If $a_n \geq 0$, and $f(x)$ is a continuous function defined on $x \in [1, \infty)$ with
  (i) $f(n) = a_n$
  (ii) $f$ decreasing
Then
\[ \sum_{n=1}^{\infty} a_n \text{ converges } \iff \int_{1}^{\infty} f(x) \, dx \text{ converges} \]  \hspace{1cm} (1)

(This Theorem follows from equation (2) below.)

Notes: 1. Since convergence depends only on the behaviour of the tail of the series, the starting values do not have to be \( x = 1 \) or \( n = 1 \).
2. To apply this test, do not forget to check conditions (ii)!
3. When (i,ii) apply (with \( a_n \geq 0 \)), be able to deduce that
\[ \int_{N+1}^{\infty} f(x) \, dx \leq \sum_{n=N+1}^{\infty} a_n \leq \int_{N}^{\infty} f(x) \, dx \]  \hspace{1cm} (2)

\( \triangleright p\text{-Series.} \) The series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) \begin{align*}
\{ & \text{converges if } p > 1 \\
\{ & \text{diverges if } p \leq 1 
\end{align*}

\( \triangleright \) **Estimating series using integrals.** If the integral test applies (and the series converges), you can approximate the series
\[ S = \sum_{n=1}^{\infty} a_n \text{ by } S_N = \sum_{n=1}^{N} a_n \]
and estimate the size of the error \( S - S_N \) using equation (2):
\[ S - S_N = \sum_{n=N+1}^{\infty} a_n \leq \int_{N}^{\infty} f(x) \, dx \]  \hspace{1cm} (3)

(You can actually get a much better estimate using the left hand side of equation (2) as well, but the book does not do this and we’ll stick to the book, keep things simple.)

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**PROBLEMS**

**DAY 24: Series - geometric, telescoping**

1. Evaluate the following series or determine that they diverge.
   
   (a) \( 1 + 0.4 + 0.16 + 0.064 + \ldots \)
   (b) \( \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n \)
   (c) \( \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \)
   
   (d) \( \sum_{n=2}^{\infty} 0.4^n \)
   (e) \( \sum_{n=2}^{\infty} 2^n \)
   (f) \( \sum_{n=2}^{\infty} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} \right) \)
   
   (g) \( \sum_{n=4}^{\infty} 5^{-n}3^{-n}4^n \)
   (h) \( \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \)
   (i) \( \sum_{n=1}^{\infty} (\tan^{-1}(n+2) - \tan^{-1} n) \)
   
   (j) \( \sum_{n=2}^{\infty} \frac{2n^2}{n^2 - 2n + 1} \)
   (k) \( \sum_{n=-3}^{\infty} \frac{1}{(-5)^n} \)

2.
DAY 25: Series - basic properties, divergence test, tail end of a series

1. State what conclusion, if any, may be drawn from the divergence test.

(a) \( \sum_{n=1}^{\infty} ne^{-n} \)  
(b) \( \sum_{n=100}^{\infty} \frac{4^n}{3^n + 3} \) 
(c) \( \sum_{j=2}^{\infty} \frac{3^j}{4^j + 3} \)  
(d) \( \sum_{l=8}^{\infty} \frac{\sqrt{l}}{\sqrt{1 + \sqrt{l}}} \)  
(e) \( \sum_{k=0}^{\infty} \frac{2k}{3k^2 + 1} \)  
(f) \( \sum_{j=-2}^{\infty} \frac{2j^2}{3j^2 + 1} \)  
(g) \( \sum_{n=1}^{\infty} \tan^{-1} n \)  
(h) \( \sum_{n=1}^{\infty} \sqrt[3]{2} \)

2. Fill in the blanks using either may or must.

(a) A series with summands tending to 0 may converge.

(b) A series that converges must have summands that tend to zero.

(c) If a series diverges, then the summands must not tend to 0.

(d) If a series diverges, then the Divergence Test may succeed in proving the divergence.

(e) If \( \sum_{n=10}^{\infty} a_n \) diverges, then \( \sum_{n=1000}^{\infty} a_n \) may diverge.

3. For each of the following series, determine whether they converge or diverge. If they converge, find their value. In each case, state which of the basic properties listed above you used.

(a) \( 3 + 2 + \frac{4}{3} + \frac{8}{9} + \ldots \)  
(b) \( \sum_{n=1}^{\infty} \frac{7^{n-1}}{(-9)^n} \)  
(c) \( \sum_{n=3}^{\infty} (5^{-n} + 2 \cdot 3^{-n}) \)

4. The harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is known to diverge.

(a) Explain why \( \sum_{n=1}^{\infty} \frac{1}{n+1} \) diverges.

(b) Does \( \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \) converge or diverge? If it converges, find its value. Why does your result not contradict basic property (iii) listed above?

(c) Does \( \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{2^n} \right) \) converge or diverge? Explain.

5. (a) Write the number 0.\(0\overline{22222}\ldots\) as a geometric series. Then evaluate that series to express the number as a ratio of integers.

(b) Use series to show that 0.\(1\overline{0}\ldots\).

6. Find the values of \( x \) for which the series \( \sum_{n=1}^{\infty} \frac{x^n}{3^n} \) converges. Find the sum of the series for those values of \( x \).

7. (a) Find the value of \( r \) if \( \sum_{n=2}^{\infty} r^n = 2 \)

(b) Find the value of \( c \) if \( \sum_{n=2}^{\infty} (1 + c)^{-n} = 2 \)
DAY 26: Integral test.

1. §12.3, # 2

2. Use the Integral Test to determine whether the series converges or diverges. Before you apply the test, be sure that the hypotheses are satisfied.

   (a) $\sum_{n=1}^{\infty} e^{-n}$
   (b) $\sum_{m=2}^{\infty} \frac{1}{m^2 + 4}$
   (c) $\sum_{k=4}^{\infty} \frac{1}{\sqrt[5]{k}}$
   (d) $\sum_{n=100}^{\infty} \frac{e^n}{(1 + e^n)^2}$
   (e) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
   (f) $\sum_{n=3}^{\infty} n 2^{-n}$

3. Use the known facts about p-series to determine which of the following series converge: §12.3: 9,10,11,12,13,14

4. Use the integral estimation test to estimate $\sum_{n=1}^{\infty} 1/n^6$ to within an error of $10^{-6}$.

5. Use the integral estimation test to estimate $\sum_{n=1}^{\infty} 1/n^2$ to within an error of 0.05. Compare with the graph of the partial sums shown below right. The summands $a_n$ are also plotted below left, for reference.