TOPICS

9. Volumes of more general objects

Reading: §6.2, Examples 7-9

- **Volumes of a solid that is not generated by rotation.** Follow the same steps (i-iv) as before:
  (i) Visualize the solid. Visualize one arbitrary slice of thickness $\Delta x$ or $\Delta y$, whatever appropriate.
  (ii) Find the volume of each slice in (i) by finding area of crosssection. This yields
    \[ \Delta V = A(x)\Delta x \quad \text{or} \quad \Delta V = A(y)\Delta y. \]
  (iii) Obtain the total volume by the sum of volumes of slices, in the limit as the slice thickness $\to 0$. This yields an integral representation of the volume of the form
    \[ \int_a^b A(x)\,dx \quad \text{or} \quad \int_c^d A(y)\,dy. \]
  
  Note: This simply says that the volume is the average cross-sectional area times width!
  
  (iv) Evaluate the integral if possible. The only thing that is different from the case of solids of revolution is the formula you use in (ii).

10. Mean Value Theorem

Reading: §4.2, page 216 and Examples 3-5 (SKIP proof)

- **Our motivation.** This is a good place to introduce the MVT (without proof) since we can motivate it by what follows, namely deriving the formula for arclength. We will use the MVT again later when we discuss the error of Taylor polynomial approximations.
  
  - **Mean Value Theorem:** If $f$ is a function that is continuous on $[a, b]$ and differentiable on $(a, b)$ then there is a number $c \in (a, b)$ such that
    \[ f'(c) = \frac{f(b) - f(a)}{b - a} \]
    (Understand corresponding picture.) When using the conclusions of the MVT, make sure to check that all conditions hold!
  
  - **Important application:** bounding function values in terms of derivative (see example 5)

11. Arclength

Reading: §9.1 (up to p 564 middle)

- **Derivation of formula for arclength.** Understand the derivation of the following, and where we need the MVT!
  
  The length of the curve $y = f(x), x \in [a, b]$ is $\int_a^b \sqrt{1 + f'(x)^2} \,dx$
  
  The length of the curve $x = g(y), y \in [c, d]$ is $\int_c^d \sqrt{1 + g'(y)^2} \,dy$
  
  In Calculus III we will consider lengths of parametric curves.
12. Work

Reading: §6.4, examples 1-4

○ Main facts. If a body is moved a distance $d$ by a constant force $F$ the work performed is

$$W = F \cdot d$$

If a body is moved from $x = a$ to $x = b$ by a variable force $F(x)$ we can approximate the force by a constant force $F(x_j)$ on small sections of length $\Delta x$. The work done to move the body on that section is approximately

$$F(x_j)\Delta x$$

The approximation is better the smaller the length $\Delta x$ is. The total work equals the sum of the work on all pieces in the limit as the number of pieces goes to infinity, that is, an integral:

$$W = \lim_{n \to \infty} \sum_{j=1}^{n} F(x_j)\Delta x = \int_{a}^{b} F(x) \, dx$$

Alternatively, we can think of it this way: The total work is the average force times the total distance moved!

○ Units and other important facts.
  - Unit of force=1 lb (pound) or 1 N (Newton)
  - Unit of work=1 ft-lb (foot-pound) or 1 Nm (Newton-meter)=1 J (Joule)
  - Unit of mass=1 kg (kilogram)
  - Force = mass $\times$ acceleration
  - Acceleration due to gravity on Earth’s $g \approx 9.8 \text{m/s}^2 \approx 32 \text{ft/s}^2$
  - Stretching a spring from equilibrium by length $x$ takes a force $F(x) = kx$ (in linear regime)

○ Sample problems of interest.
  - Work done in stretching a spring.
  - Work done in lifting an object whose weight is changing in time.
  - Work done in pumping water out of a basin.
  - Work done in lifting an object whose weight is changing in time and subject to a changing force (such as when launching a satellite into orbit).

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PROBLEMS

DAY 15: Volumes of more general objects.

1. The base of a solid $S$ is the disk $x^2 + y^2 \leq 25$. For each $k \in [-5, 5]$, the plane through the line $x = k$ and perpendicular to the $xy$-plane intersects $S$ in a square. Find the volume of $S$.

2. The base of a solid $S$ is the elliptical region $9x^2 + 4y^2 \leq 36$. Cross-sections perpendicular to the $x$-axis are isosceles right triangles with hypotenuse in the base. Find the volume of $S$.

3. The base of a solid $S$ is the triangle in the $x$-$y$ plane with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Cross-sections perpendicular to the $y$-axis are equilateral triangles. Find the volume of $S$. 
DAY 16: Mean value Theorem. Arclength.

1. Mean value theorem: (a-d) page 219 # 7, 11, 12, 24.

2. Find the arc length of the curves described by the following functions
   (a) $f(x) = 2 + x^{3/2}, x \in [1, 4]$
   (b) $f(y) = (y - 1)^{3/2}, y \in [2, 5]$
   (c) $f(x) = \cosh(x) + 7, x \in [0, 1]$

3. Find the arc length of the graph of $y^4 - 16x + 8/y^2 = 0$ between $P = (9/8, 2)$ and $Q = (9/16, 1)$.

DAY 17: Work.

1. How much work is done in lifting a 40-kg sandbag to a height of 1.5 m? (p373,# 1)

2. p 373, # 5 (Given force defined piecewise, find work)

3. A spring is stretched 2 in beyond its equilibrium position. If the force required to maintain it in its stretched position is 60 lb, how much work has been done?

4. p 373, # 11 (A spring problem. Compare work done by stretching different portions of spring.)

5. p 373, # 13 (Lift a rope to the top of a building)