This pdf file lists daily homework problems, consisting of

Level 1 problems: strengthen the basics (do these first)

Level 2 problems: apply the basics and build understanding

Mixed Review: mixed review

Quizzes and Exams will consist mainly of problems like Level 2 and Mixed Review, with some Level 1 possible. You are responsible for all. You must work on all problems on a daily basis for a chance to succeed in this class. Additional practice problems with answers are given by the odd problems in the book.

Solve all problems without using a calculator, unless specified, as you will not be able to use a calculator on quizzes and exams. You can get any help you are comfortable with, but ultimately, you need to write out the complete solutions on your own, without referring to notes/book/web/tutor.

**HOMEWORK DAY 1 — Tangent lines. The limit \( \lim_{x \to a} f(x) \). Computing limits using tables.**

**Reading:** §1.4: Introduction and Example 1. §1.5: Introduction and Example 1.

**Level 1:** §1.1: 1, 2, 26 (simplify answer), 28, 30, 55, 77, 78. Review: even/odd fcns; Pascal’s triangle.

**Level 2:**

1. (a) Explain what the following means and illustrate with a sketch \( \lim_{x \to a} f(x) = L \).

   (b) Describe several ways in which a limit fails to exist. Illustrate with sketches.

   (c) Explain with a sketch why \( \lim_{x \to 2} x^2 = 4 \).

2. Consider the two functions \( f(x) = \frac{x^2 - x}{x - 1} \) and \( g(x) = x \).

   (a) Are the two functions equal? Explain.

   (b) Sketch a graph of both functions.

   (c) Use your sketch to find \( \lim_{x \to 1} f(x) \) and \( \lim_{x \to 1} f(x) \). Illustrate the limit in your sketch.

3. **Calculator.** Use a table of values to determine the following limits \( \lim_{x \to a} f(x) \). Choose the values of \( x \) near \( a \) in your table and remember to use values approaching \( a \) from either side. If the limit does not exist, explain why not.

   (a) \( \lim_{x \to 1} x^3 \)  

   (b) \( \lim_{x \to 1} \frac{(1 + x)^3 - 1}{1} \)  

   (c) \( \lim_{x \to 1} \frac{1}{x - 1} \)  

   (d) \( \lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x} \)

4. Given a function \( f(x) \) and a point \( x = a \)

   (a) Describe in words how the slope of the tangent line of \( f \) at \( x = a \) is defined. Illustrate with a sketch.

   (b) Use your sketch to find a mathematical expression for the slope of the tangent line as a limit.

5. Let \( f(x) = \sin(x) \).

   (a) Sketch a clearly labeled graph of \( f(x) \), using a 1-1 scale.

   (b) Explain why the slope of the tangent line to the graph of \( f \) at the origin \( (x = 0) \) is given by the limit

   \[
   \lim_{x \to 0} \frac{\sin x}{x} .
   \]

   (c) **Calculator.** Approximate the limit using a table of values.

   (d) Use your result in (c) to find an equation for the tangent line to \( f \) at the origin, and add a graph of it to your sketch in (a).

**Mixed Review:**

R1.1. Diagnostic test A (page xxiv in Stewart): all
R1.2. Sketch the graphs of the following functions, one graph per window. Each graph should be clearly labelled, including the axes, and any important points on the graphs, such as intercepts, vertices, local maxima. Note: when graphing polynomials, see if you can factor the function first, then use the roots and behaviour at infinity to sketch the graph.

(a) \( f(x) = 1 - x^2 \)  
(b) \( f(x) = x - x^2 \)  
(c) \( f(x) = x - x^3 \)  
(d) \( f(x) = \sin(3x) \)  
(e) \( f(x) = \cos(\pi/2) \)

R1.3. Mark all answers that are correct. The domain of \( g(t) = \sqrt{3-t} - \sqrt{2+t} \) is:

- all \( t \) with \( t \leq 3 \) and \( t \geq -2 \)
- all \( t \) with \( t \leq 3 \) or \( t \geq -2 \)
- all \( t \in (-2, 3) \)
- all \( t \in [-2, 3] \)
- all \( t \in (-\infty, 3) \cup [-2, \infty) \)
- all \( t \in (-\infty, -2] \cup [3, \infty) \)

**HOMEWORK DAY 2a — Finite and infinite limits. Onesided limits.**

**Reading:** §1.5: Examples 6-10. §1.6: Examples 2-9.

**Level 1:** §1.5: 1,4,5,8. §1.6: 6,11,12,15,17.

**Level 2:**

1. Find the following finite or infinite limits. If the limit does not exist, explain why not.

(a) \( \lim_{x \to -2} (2x + x^2) \)  
(b) \( \lim_{h \to 1} (2 - h/2) \)  
(c) \( \lim_{x \to 1^+} \frac{1}{x - 1} \)  
(d) \( \lim_{x \to 3^-} \frac{x + 2}{x + 3} \)  
(e) \( \lim_{x \to 1} \frac{1}{x - 1} \)  
(f) \( \lim_{x \to 1} \frac{1 - 2x}{(x - 1)^2} \)  
(g) \( \lim_{x \to \pi} \frac{1 + \cos x}{1 - x} \)  
(h) \( \lim_{x \to \pi} \frac{1 - x}{1 + \cos x} \)  
(i) \( \lim_{t \to 1} \frac{1 - t^2}{1 - t} \)  
(j) \( \lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x} \)  
(k) \( \lim_{x \to 1} \frac{(1 + x)^2 - 1}{x} \)  
(l) \( \lim_{x \to 1} \frac{(1 + x)^4 - 1}{x} \)

2. Let \( f(x) = \begin{cases} 
1 + x & \text{if } x < -1 \\
2 - x & \text{if } -1 \leq x < 1 \\
-1 & \text{if } x > 3 
\end{cases} \)

(a) Find the following limits or determine they do not exist. Show all relevant work.

\( \lim_{x \to -2} f(x), \lim_{x \to -1} f(x), \lim_{x \to 1} f(x), \lim_{x \to 3} f(x). \)

(b) Sketch a graph of the function.

(c) State the values of all \( a \) for which \( \lim_{x \to a} f(x) \) exists.

3. Consider the function \( g(x) = \frac{x^2 - 4}{x - 2} \).

(a) Find \( \lim_{x \to 2} g(x) \).

(b) Sketch a graph of \( g(x) \).

(c) Explain why the limit in (a) is the slope of tangent line of \( f(x) = x^2 \) at \( x = 2 \). Illustrate with a figure.
4. Consider the function \( f(x) = \frac{\sin x}{x} \).
   (a) State the domain of \( f \).
   (b) Show \( f \) is even.
   (c) Find the values of \( x \) for which \( f(x) = 0 \).
   (d) What happens to the values of \( f \) as \( x \to \pm\infty \)? Why?
   (e) State the limit \( \lim_{x \to 0} f(x) \) (refer to HW Day 1).
   (f) Use your information above to sketch a graph of \( f(x) \).

Mixed Review :
R2a.1. Diagnostic test B (page xxvi in Stewart): all
R2a.2. Sketch a graph of the given functions. Use the graph to solve the following inequalities.
   (a) \( f(x) = x^3(x - 1), \quad x^3(x - 1) \geq 0 \)
   (b) \( f(x) = x^3 - x, \quad x^3 \leq x \)
   (c) \( f(x) = x^3 - 5x^2 + 4x, \quad x^3 - 5x^2 + 4x < 0 \)
   (d) \( f(x) = 1/x, \quad 1/x < 2 \)

**HOMEWORK DAY 2b — Definition of \( f'(a) \). Rates of change.**

**Reading:** §2.1: Introduction and examples 1,2. page 108 and examples 6,7.

**Level 1:** §2.1: 5,17,18,20,33,35.

**Level 2:**

1. Consider \( f(x) = x^3 - 3x \).
   (a) Use the definition to find \( f'(a) \). For which values of \( a \) is \( f \) differentiable?
   (b) Find the equation for the tangent line to the graph of \( f(x) \) at \( a = 2 \).
   (c) Sketch the graph of \( f \) and its tangent line.

2. Consider \( f(x) = \sqrt{x + 1} \).
   (a) Use the definition to find \( f'(a) \). For which values of \( a \) is \( f \) differentiable?
   (b) Find the equation for the normal line to the graph of \( f(x) \) at \( a = 1 \).
   (c) Sketch the graph of \( f \) and its normal line at \( a = 1 \).

3. §2.1: 34.

4. A cylindrical tank initially holds 100,000 gallons and is being drained from the bottom over the interval of 1 hour. The amount of gallons of water \( V \) remaining in the tank after \( t \) minutes is given by
   \[ V(t) = 100,000 \left( 1 - \frac{t}{60} \right)^2, \quad t \in [0, 60]. \]
   (a) What is the meaning of the derivative \( V'(t) \)? What are its units?
   (b) For \( t > 0 \), is \( V'(t) \geq 0 \) or \( \leq 0 \)? Explain, using only your answer in (a).
   (c) In practical terms, what does it mean to say that \( V'(10) \approx -2777.8? \)
   (d) Sketch a graph of \( V(t) \). What is a geometrical meaning of the derivative \( V'(t) \)?
   (e) At what time is \( |V'(t)| \) largest? Explain physically why that makes sense.

5. §2.1: # 49

6. A particle moving along a horizontal line has position \( s(t) \), where \( s \) is measured in meters, \( t \) in seconds.
   (a) What is the meaning of the derivative \( s'(t) \)? What are its units?
   (b) What is the meaning of the derivative \( s''(t) = (s')'(t) \)? What are its units?

7. §2.1: # 11

**Mixed Review :**
R2b.1. Diagnostic test C (page xxvii in Stewart): all
R2b.2. **Note 1:** \( \sqrt{x} \) is only defined for \( x \geq 0 \)
   **Note 2:** \( \sqrt{x} \) is always positive (sketch \( y = \sqrt{x} \)!)
Note 3: \( \sqrt{x^2} = |x| \) (which is always positive)

Note 4: The solution to \( x^2 > 9 \) is \(|x| > 3\). That is, all positive and negative \( x \) values with magnitude bigger than 3: \( x > 3 \) or \( x < -3 \).

Simplify:
(a) \( \sqrt{(-2)^2} = \)  
(b) \( \sqrt{9} = \)  

Solve the following, where \( a \) is a positive constant. Make sure to correctly use the words “AND” or “OR” when appropriate.

(c) \( x^2 = 4 \)  
(d) \( x^2 > 4 \)  
(e) \( x^2 > a^2 \)  
(f) \( x^2 < a^2 \)  
(g) \( |x| > a \)  
(h) \( |x| < a \)

**HOMEWORK DAY 3 — The function \( f'(x) \) (graphs, definition)**

**Reading:** §2.2: Examples 1-3.

**Level 1:** §2.2: 3,4,5,8,15.

**Level 2:**

1. §2.2: 6,7,9 (given graph of \( f \), graph \( f' \))
2. Consider the function \( f(x) = mx + b \), where \( m,b \) are constants.
   (a) Use the definition to find \( f'(x) \) (show all steps!)
   (b) Sketch a graph of \( f \) and \( f' \).
3. Use the definition to find the derivative of \( f(x) = \frac{1}{\sqrt{x}} \).
4. Use the definition to find the derivative of \( g(t) = \frac{1 - 2t}{t + 3} \).

**Mixed Review :**

R3.1. Diagnostic test D (page xxiv in Stewart): all

R3.2. Find the domain of the following functions. Then sketch the graph of \( f \). Here \( c \) and \( A \) are positive constants.  
(a) \( f(x) = \sqrt{c^2 - x^2} \)  
(b) \( f(x) = A\sqrt{c^2 - x^2} \)

R3.3. Sketch the graphs of each of the following functions,

(a) \( v(r) = r^2(r_o - r) \), \( r_o > 0 \). 
(b) \( f(x) = x^2 - r^2 \), \( r > 0 \) 
(c) \( f(t) = \sqrt{t} \)  
(d) \( f(\theta) = \tan(\theta) \)

**HOMEWORK DAY 4a — Rules for differentiation (c, \( x^p \), \( cf \), \( f \pm g \))**

**Reading:** §2.3: pp 126–130, including examples 1-5.

**Level 1:** §2.3: 3,5,7,9,15,16,19,21,37

**Level 2:**

1. §2.3: 6,14,20,33,38,41
2. §2.3: 81
3. (a) State the mathematical definition of the derivative \( f'(x) \). Include an illustrative sketch. 
   (b) Explain, using another illustrative sketch, why \( f'(x) \) can also be defined by

\[
    f'(x) = \lim_{h \to 0} \frac{f(x) - f(x - h)}{h}
\]

(c) Assume \( f \) is even, that is, \( f(x) = f(-x) \). Use the definition of the derivative (given in part a or b above) to show that \( f'(x) \) is odd, that is, that \( f'(-x) = -f'(x) \).

(d) Give an example of an even function \( f \), and confirm that its derivative is odd.

**Mixed Review :**

R4a.1. Sketch the graphs of the following functions, one graph per window, all clearly labelled.

(a) \( g(x) = \sqrt{x - 5} \)  
(b) \( f(x) = \sqrt{x^2} \)
(c) $f(x) = |x + 2|$  
(d) $f(x) = \frac{1}{2}(1 - \cos(\pi x))$

**HOMEWORK DAY 4b — Rules for differentiation: Products, quotients**

**Reading:** §2.3: Examples 6, 7, 9-12

**Level 1:** §2.3: 25, 26, 28, 29, 30, 31, 32

**Level 2:**
1. §2.3: 24
2. §2.3: 8, 44
3. Find all points on the graph of $y = t/(t^2 - 1)$ where the tangent line is horizontal.
4. §2.3: 57
5. §2.3: 72
6. §2.3: 74

**Mixed Review:**
R4b.1. (a) State the mathematical definition of the derivative $f'(a)$. Draw a sketch that illustrates its geometrical significance. Discuss a second way of interpreting this number.
   (b) Use the definition to find $f'(x)$ for $f(x) = \sqrt{2x - 1}$.
   (c) Use the definition to find $f'(x)$ for $f(x) = 2x^4$.

R4b.2. A manufacturer produces bolts of a fabric with a fixed width. The cost of producing $x$ yards of this fabric is $C = f(x)$ dollars.
   (a) What is the meaning of the derivative $f'(x)$? What are its units?
   (b) In practical terms, what does it mean to say that $f'(1000) = 9$?

**HOMEWORK DAY 5 — Rules for differentiation: Trigonometric functions**

**Reading:** §2.4: Examples 1-3.

**Level 1:** §2.3: 58, §2.4: 1, 5, 6, 9, 21, 49.

**Level 2:**
1. Find the derivatives of the following functions
   (a) $f(x) = 5x^2 - 3 \cos x$
   (b) $g(s) = \sqrt{2s} \sin s$
   (c) $y = \frac{1 - \sec t}{\tan t}$
   (d) $y = u(a \cos u + b \cot u)$ where $a, b$ are constants

2. §2.4: 35

3. A ladder 20 ft long rests against a vertical wall. Let $\theta$ be the angle between the top of the ladder and the wall and let $x$ be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does $x$ change with respect to $\theta$ when $\theta = \pi/6$?

4. §2.4: 38

5. Find $\frac{d^3}{dx^3}(x \cos x)$

**Mixed Review:**
R5.1. §2.3: 2, 10, 13, 25, 27, 31, 36

R5.2. The following limits represent the slope of the line tangent to the graph of $y = f(x)$ at some point $x = a$, for some function $f$. In each case, state $f$ and $a$.

(a) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$  
(b) $\lim_{h \to 0} \frac{1+\frac{h}{2} - 1}{h}$  
(c) $\lim_{x \to 0} \frac{\sin(x)}{x}$  
(d) $\lim_{x \to 0} \frac{\cos(x) - 1}{x}$

Explain why the limit in (d) must equal 0, using only properties of $f$, not a formula for the derivative.
HOMEWORK DAY 6 — Rules for differentiation: Exponential function

Reading: §6.2: pages 391–396, examples 1.
Level 1: §6.2: 2,5,15,16,22
Level 2:
1. Sketch the graphs of the following functions
   (a) \( y = 2^x, \ y = e^x, \ y = 10^x, \ y = 2^{-x} \) (sketch all on one common screen)
   (b) \( y = e^{-x} \)
   (c) \( y = -e^{-x} + 1 \)
   (d) \( y = e^{-x+1} \)
2. §6.2: 18
3. Find the derivatives of the following functions
   (a) \( y = e^2 - \sqrt{x/2} \)
   (b) \( k(r) = -e^r + r^e \)
   (c) \( y = (x^3 - \frac{1}{x})e^x \)
   (d) \( y = \frac{e^t}{1-e^t} \)
4. Find equations for the lines tangent and normal to the curve \( y = \frac{e^x}{x} \) at the point \((1,e)\).
5. Find all the points on the graph of \( y = e^x \sin x \) at which the slope is zero.

Mixed Review:
R6.1. Consider the function \( v(r) = r^2(r_0 - r) \), where \( r_0 > 0 \) is a constant.
   (a) Without using any calculus, sketch a graph of this cubic polynomial.
   (b) Find the coordinates of the point at which the graph has a horizontal tangent.
R6.2. The graph shows the volume \( V \) of water in a tank \( t \) minutes after some start time. What is the meaning of the derivative \( V'(t) \)? What are its units? When is the volume increasing? When is it decreasing? At what rate is water entering the tank after 2 minutes? Sketch a graph of the rate of change of the volume of water in the tank. The graph must be clearly labeled.

R6.3. Use the definition of the derivative to find \( f'(x) \) for \( f(x) = \cos(x) \).

HOMEWORK DAY 7 — Rules for differentiation: Chain Rule

Reading: §2.5: Introduction and examples 1-6. §6.2: examples 2,3.
Level 1: §2.5: 7,13,17,21,36,51. §6.2: 35,37,43
Level 2:
1. Find the derivatives of the following functions §2.5: 7,12,14,16,34, §6.2: 36,42,48
2. §2.5: 65
3. §2.5: 76 (harmonic motion)
4. Find the $x$-coordinates of all points in the interval $[0, 2\pi]$ on the graph of $h(x) = 3\cos^2 x + 3\cos x$ at which the tangent line is horizontal.

Mixed Review:
R7.1. Find the first and second derivatives of $f(x) = \frac{x^2}{a + x}$ where $a$ is a constant. Solve $f'(x) = 0$, $f''(x) = 0$.
R7.2. If $g$ is a differentiable function, find an expression for the derivative of each of the following functions
   (a) $y = xg(x)$
   (b) $y = \frac{x}{g(x)}$
   (c) $y = g(x)
R7.3. Sketch the graph of $y = 1 - \frac{1}{2}e^{-x}$.

HOMEWORK DAY 8 — Implicit Differentiation

Reading: §2.6: Examples 1,2.
Level 1: §2.6: 1,13,14,28. §6.2: 38,54

Level 2:
1. Find $dy/dx$, where $y$ is defined implicitly by the following equations
   (a) $x^2 + 2xy - y^2 = 6$
   (b) $\frac{1}{x} + \frac{1}{y} = 1$
   (c) $x^4 + y^4 = 1$
   (d) $y\cos(x) = x^2 + 2y^2$
   (e) $\tan(x/y) = x + 2y$
2. §2.6: 21
3. §2.6: 25 (Find equation for the line tangent to given point)
4. §6.2: 53

Mixed Review:
R8.1. Find the derivatives of the following functions. Simplify your answer.
   (a) $f(x) = \cos(x^3)$
   (b) $g(x) = \cos^3 x$
   (c) $g(t) = (2t - 5)\tan(t)$
   (d) $h(s) = (3s + 2)^8(s^2 + 3)^6$
R8.2. §2.4: 29 (derivatives of $H(\theta) = \theta \sin(\theta)$)
R8.3. Evaluate the limits: (a) $\lim_{x\to-5} \frac{x^2 - 25}{x + 5}$
   (b) $\lim_{s\to1} \frac{s^2}{(1 - s)^2}$
R8.4. Find $dy/dx$ where
   (a) $y = \tan(\sqrt{x})$
   (b) $y = \sqrt{\tan(x)}$

HOMEWORK DAY 11 — Rates of Change

Level 1: §2.1: 43, §2.2: 13,44 §2.7: 2, 5, 11, 15.

Level 2:
1. §2.7: 4
2. §2.7: 20 (gravitational force)
3. §2.1: 50
4. The volume of a right circular cone is $V = \frac{\pi r^2 h}{3}$, where $r$ is the radius of the base and $h$ is the height.
   (a) Find the rate of change of the volume with respect to the height if the radius is constant.
   (b) Find the rate of change of the volume with respect to the radius if the height is constant.
5. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. A model for the water depth $D$ at Hopewell Cape as a function of time $t$ (in hours after midnight) is given by
   $$D(t) = 7 + 5\cos\left(\frac{2\pi(t - 6)}{12}\right)$$
(a) Without using calculus, answer the following: What is the depth at high tide? What is it at low tide? In a 24 hour period, \( t \in [0, 24] \), when is it highest? Sketch a graph of \( D(t) \) for \( t \in [0, 24] \).

(b) Find a function for the rate of change of the water depth. In a 24 hour period, \( t \in [0, 24] \), when does the depth change the fastest?

Mixed Review:

R11.1. Find the points on the graph of \( P(I) = \frac{100I}{I^2 + I + 4} \) at which the tangent line is horizontal.

R11.2. Find \( \frac{dy}{dx} \) where \( \sin(xy) = x^2 - y \).

R11.3. The following limits represent the slope of the line tangent to the graph of \( y = f(x) \) at some point \( x = a \), for some function \( f \). In each case, state \( f \) and \( a \).

\[
\begin{align*}
\text{(a)} & \quad \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\
\text{(b)} & \quad \lim_{h \to 0} \frac{1 + h - 1}{h} \\
\text{(c)} & \quad \lim_{x \to 0} \frac{\sin(x)}{x} \\
\text{(d)} & \quad \lim_{x \to 0} \frac{\cos(x) - 1}{x}
\end{align*}
\]

Explain why the limit in (d) must equal 0, using only properties of \( f \), not a formula for the derivative.

HOMEWORK DAY 12 — Related Rates

Level 1: §2.8: 3,7,11,15.

Level 2:

1. §2.8: # 4 (rectangle)
2. §2.8: # 5 (cylindrical tank)
3. §2.8: # 12 (snowball)
4. §2.8: # 14 (two ships)
5. §2.8: # 20 (boat pulled into dock)
6. §2.8: # 27 (growing sand pile)
7. §2.8: # 35 (two resistors connected in parallel)

Mixed Review:

R12.1. §2.1: 11 (distance/velo)
R12.2. §2.1: 51 (interpret derivatives, units)

HOMEWORK DAY 13 — Linear Approximation

Level 1: §2.9: 1,5,41

Level 2:

1. §2.9: 4
2. Linear approximations of functions. Show that for any real number \( k \), \((1 + x)^k \approx 1 + kx\) for small \( x \). Estimate \((1.02)^{0.7}\) and \((0.9)^{-0.3}\).
3. Linear Extrapolation. The Table in page 175, # 25, shows the world population every 10 years during the last century.
   (a) Use the last two entries to estimate the slope of the function \( P(t) \) at \( t = 2000 \) (ie, the rate of change or the population with respect to time).
   (b) Use your result in (a) to find the linear approximation of \( P(t) \) at \( t = 2000 \).
   (c) Use the linear approximation to estimate the population at \( t = 2015 \).
   (d) Go online and find the actual population in 2015 and compare with your estimate in (c). Is your estimate an under- or over-estimate?
4. Approximating measurement errors. The circumference \( c \) of a sphere was measured to be 84 cm with a possible error of at most 0.5 cm (in absolute value). Approximate the maximum error using this measurement of \( c \) to compute
(a) the surface area of the sphere  
(b) the volume of the sphere  
(Hint: in (a) the first step is to write surface area in terms of \( c \). Then write an approximation of \( \Delta S \) in terms of \( \Delta c \). Similarly for (b).)  
Also compute the exact maximum error and compare to your approximation.  
Note: In practice, the approximation is much easier to obtain and is sufficiently good, since measurement errors are not precise in the first place.  

5. **Approximating changes.** Approximate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemispherical dome with diameter 50 m. Use linear approximations. Also compare the exact amount of paint needed and compare to the linear approximation. In practice, the coat of paint will not be precisely 0.05cm thick so the approximation gives a good estimate.  

6. **Approximating changes.** Approximate the volume of a thin cylindrical shell with height \( h \), inner radius \( r \), and thickness \( \Delta r \). Use linear approximations. Also find the exact volume and compare.  

7. Find the linear approximation \( L(x) \) of \( f(x) = e^x \) at \( x = 0 \). Sketch a graph of both \( f(x) \) and \( L(x) \).  

8. Find the linear approximation \( L(x) \) of the function \( y(x) \) defined implicitly by \( xe^y + ye^x = 1 \) at \( x = 0 \).  

**Mixed Review:**  
R13.1. Sketch the graphs of the following functions, one graph per window. Each graph should be clearly labelled, including the axes, and any important points on the graphs, such as intercepts, vertices, local maxima.  

(a) \( f(x) = 2 - 0.4x \)  
(b) \( f(t) = 2t + t^2 \)  
(c) \( g(x) = \sqrt{x - 5} \)  
(d) \( f(x) = \sqrt{x^2} \)  
(e) \( f(x) = |x + 2| \)  
(f) \( f(x) = |x| - x \)  
(g) \( f(x) = \begin{cases} x + 2 , & x \leq -1 \\ x^2 , & x > -1 \end{cases} \)  
(h) \( f(x) = \begin{cases} x + 9 , & x < -3 \\ -2x , & |x| \leq 3 \\ 2 - x^2 , & x > 3 \end{cases} \)  
(i) \( f(x) = \sqrt{1 - x^2} \)  
(j) \( f(x) = |1 - x^2| \)  
(k) \( f(x) = \sin(3x) \)  
(l) \( f(x) = \frac{1}{2}(1 - \cos(\pi x)) \)  
(m) \( f(x) = \tan(x) \)  
(n) \( f(x) = H(x) \) where \( H(x) \) is as defined in p 44,§ 57  
(o) \( f(x) = H(x - 1) \)  

R13.2. Find the points on the curve \( y = \frac{\cos x}{2 + \sin x} \) in \([0, 2\pi]\) at which the tangent is horizontal.  

**HOMEWORK DAY 14 — Definitions; find extrema by inspection; critical numbers**  

**Level 1:** §3.1: 5,15,22,23,25,27,32,34,36,40,41,59  

**Level 2:**  
1. Find the domain and absolute maximum of \( f(x) = \sqrt{r^2 - x^2} \). Include a graph of the function to justify your answer.  
2. Sketch the graph of \( f(t) = \cos(3\pi t)/2 + 1 \) and state its absolute maximum and minimum values.  
3. §3.1: 28 (sketch piecewise and find extrema)  
4. Find the absolute maxima and minima of the following functions, clearly explaining your reasoning.  
   (a) \( f(x) = 2x^3 - 3x^2 - 12x + 1 , \ -2 \leq x \leq 3 \)
\[ f(x) = x - 2 \cos x, \quad 0 \leq x \leq 2 \]
\[ f(x) = x^2(x^2 - 1), \quad x \in [-1, 2] \]
\[ f(t) = \sqrt[3]{t}(8 - t), \quad t \in [0, 8]. \]
\[ f(t) = 2 \cos t + \sin(2t), \quad 0 \leq t \leq \pi/2 \]

5. §6.2: 68

Mixed Review:

R14.1. Find the derivatives of the following functions. Simplify your answer.
(a) \( P(R) = \frac{E^2R}{(R+r)^2} \)
(b) \( g(s) = \sqrt{2s + \frac{1}{\sqrt{2s}}} \)
(c) \( f(x) = x\sqrt{x^2 + 1} \)

R14.2. §2.7: #28 (violin string)

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**HOMEWORK DAY 15 — Derivatives and the shape of the graph**

Level 1: §3.3: 1, 8, 10, 49.

Level 2:
1. Find the absolute maxima and minima of the following functions, clearly explaining your reasoning
   (a) \( f(x) = x + \frac{1}{x}, \quad x > 0 \)
   (b) \( f(x) = ax + \frac{b}{x^2}, \quad x > 0 \) where \( a, b > 0 \)

2. For the function \( f(x) = 3x^4 - 4x^3 + 2 \)
   (i) Find the intervals where the function is increasing/decreasing and the coordinates of all local max/min.
   (ii) Find the intervals where the function is concave up/down and the coordinates at all inflection points.
   (iii) Use the above information to sketch a rough graph of the function.

3. For the function \( f(x) = \frac{2}{x^2 + 4} \)
   (i) Find the intervals where the function is increasing/decreasing and the coordinates of all local max/min.
   (ii) Find the intervals where the function is concave up/down and the coordinates at all inflection points.
   (iii) Find the limiting behaviour as \( x \) approaches \( \pm \infty \).
   (iv) Use the above information to sketch a rough graph of the function. Include all intercepts on graph.

4. §3.3, #14 (trig functions)
5. §3.3, #18 (compare 1st and 2nd derivative test)
6. §6.2, #66

Mixed Review:

R15.1. §2.9: 6 (do not use a calculator for graph), 41.
R15.2. Find the absolute maxima and minima of \( f(x) = x^{4/5}(x-4)^2 \) on the interval [-1,4] (use calculator to evaluate \( f \) at one of the critical points). Sketch a graph of the function on that interval.

R15.3. The total cost (in $) of repaying a student loan at an interest rate of \( r \% \) per year is \( C = f(r) \).
(a) What is the meaning of the derivative \( f'(r) \)? What are its units?
(b) What does the statement \( f'(10) = 1200 \) mean?
(c) Is \( f'(r) \) always positive or does it change sign?

**HOMEWORK DAY 16 — Limits at infinity. Graphing rational and other functions.**

**Level 1:**
1. §3.4: 2,4,9,11,13,50,52

**Level 2:**
1. §3.4: 10,12,14,17,18
2. Let \( f(x) = \frac{4-x}{3+x} \).
   (a) Find the intercepts of \( f \) and all its asymptotes. Also find the limiting behaviour of \( f \) near its vertical asymptotes (\( \lim_{x \to a^\pm} f(x) \)).
   (b) Use the above information to sketch a guess for the graph of \( f \).
   (c) Confirm details of your guess by finding the intervals where the function is increasing/decreasing and concave up/down. Clearly mark any local extrema and inflection points in your plot.
   (d) Use algebra (long division in this case) to rewrite the function as a polynomial plus a proper quotient. Now sketch a graph of \( f \) using translations/dilations of \( y = \frac{1}{x} \). Which way of obtaining the sketch do you prefer?
3. Consider the function \( f(x) = \frac{\sin(x)}{x} \).
   (a) State the domain and all intercepts of the function. Is the function odd or even? Show it.
   (b) Find all asymptotes of the function.
   (c) Using what we learned earlier in this class, explain why
   \[ \lim_{x \to 0} f(x) = 1 \]
   (c) Sketch a graph of the function that reflects all the information you found above.
4. The reaction rate \( V \) of a common enzyme reaction is given in terms of substrate level \( S \) by
   \[ V = \frac{V_o S}{K + S}, \quad S \geq 0 \]
   where \( V_o \) and \( K \) are positive constants.
   (a) Show that \( V \) is an increasing function of \( S \). Explain why it follows that \( V \) has no absolute maximum value.
   (b) What is \( \lim_{S \to \infty} V \)?
   Determine the concavity of the graph of \( V \) (where is it concave up? where concave down?).
   (c) Sketch a graph of \( V \) as a function of \( S \) that reflects all the information you found above.

**Mixed Review :**
R15.1. §1.1: 48 (graph of functions defined piecewise)
R15.2. Find the linear approximation \( L(x) \) of \( f(x) = \tan x \) at \( x = \pi/4 \). Sketch a graph of both \( f \) and \( L \) on one screen.

R15.3. §3.7, # 9 (maximizing yield)

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**HOMEWORK DAY 17 — Graphing**

**Level 1:** §3.5: 1,22,34

**Level 2:**

For each of these problems, sketch a graph clearly showing domain, asymptotes, intercepts, local maxima/minima and inflection points.

1. §3.5: 4
2. §3.5: 15
3. §3.5: 26
4. §3.5: 43 (Hint: replace \( W/(24EI) \) by \( A \), this simplifies the formulation.)
5. §6.2: 69

**Mixed Review :**

R17.1. §6.2: 39,43,46
R17.2. Find the absolute maximum value and absolute minimum value of \( f(x) = \cos^2 x - 2 \sin x \) for \( x \in [0, 2\pi] \).
R17.3. §2.9, # 38 (flux of blood through vessel: Poiseuille’s Law)

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**HOMEWORK DAY 18**

**Optimization**

**Level 1:** §3.7: 3,7,8,9.

**Level 2:**

1. Assume that if the price of a certain book is \( p \) dollars, then it will sell \( x \) copies where \( x = 7000(1 - p/35) \). Suppose the dollar cost of producing those \( x \) copies is \( 15000 + 2.5x \). Finally, assume that the company will not sell this book for more than \$35. Determine the price for the book that will maximize profit.
2. §3.7: 10
3. §3.7: 12
4. §3.7: 16
5. §3.7: 19
6. §3.7: 21

**Mixed Review :**

R18.1. Show that \( \sqrt{1 + 2x} \approx 1 + \frac{x}{2} \) if \( x \approx 0 \). Use this result to approximate \( \sqrt{0.9} \).
R18.2. Chapter 3, Review: 2,6 (max cont fcn on closed interval)
R18.3. Sketch the graphs of the following functions, clearly showing local extrema, inflection points, and intercepts where possible.

(a) \( f(x) = x + 2 \cos(x) \), \( x \in [-\pi, \pi] \)  
(b) \( f(x) = x(x - 1)^3 \) Hint: you need to set \( f' \) and \( f'' \) to zero. For this, you need to factor, not foil. So, DO NOT FOIL. Use product rule and factor. Repeat.
R18.4. §3.1, 67 (Find max of \( v(r) = k(r_0 - r)r^2 \))
R18.5. The velocity of a wave of length $L$ in deep water is

$$v = K \sqrt{\frac{L}{C} + \frac{C}{L}}$$

where $K$ and $C$ are positive constants. What is the length of the wave that gives the minimum velocity?

R18.6. §3.7, 20

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**HOMEWORK DAY 21 — Antiderivatives. Differential equations. Initial value problems**

**Level 1:** §3.9: 1,2,5,7,9,13, 23,24,29,31,68.

**Level 2:**

§3.9: 6,10,12,16,17,29,36,38,45,47,56,62,64

**Mixed Review :**

R21.1. §6.2: 67 (find abs max/min of continuous exponential function on closed interval $[a, b]$)

R21.2. §3.7: 14. (optimization, box)

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**HOMEWORK DAY 22 — Approximating areas. Riemann sums.**

**Level 1:** §4.1: 5,7

**Level 2:**

1. §4.1: 2

2. Use the left-endpoint rule with $n = 6$ to approximate $\int_{0}^{\pi} \sin x \, dx$. Simplify your answer as much as possible without using a calculator. Draw a diagram and explain what the sum represents.

3. Evaluate the following sums.

   (a) $\sum_{k=2}^{5} \frac{2k}{k-1}$  
   (b) $\sum_{j=0}^{5} j^2 \sin\left(\frac{j\pi}{6}\right)$  
   (c) $\sum_{k=3}^{100} \frac{2}{k}$  
   (d) $\sum_{j=1}^{1000} j$

4. Use summation notation to express the sums

   (a) $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$  
   (b) $3 + 6 + 9 + 12 + 15$  
   (c) $\frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \frac{5}{11}$

5. Which is larger, $\sum_{j=1}^{N} j^2$ or $\sum_{j=1}^{N^2} j$? Explain why.

**Mixed Review :**

R22.1. Find $dy/dx$ and simplify your answer.

(a) $y = x \sin(2x)$  
(b) $y = \frac{x+1}{x^2+1}$  
(c) $xy = \tan y$

R22.2. §1.1: 49,50 (graph functions defined piecewise.)

R22.3. Find the following indefinite integrals:

(a) $\int \sqrt{\frac{5}{x}} \, dx$  
(b) $\int \frac{1 + \cos^2 \theta}{\cos^2 \theta} \, d\theta$

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**HOMEWORK DAY 23 — Definition of definite integral. Properties.**

**Level 1:** §4.2: 8,9,17,19,33,37,39,41,43,47,49.

**Level 2:**

1. §4.2: 2,8,18,20,34,40,42,48,50,52
2. The **Heaviside function** \( H \) is defined by

\[
H(t) = \begin{cases} 
0, & t < 0 \\
1, & t \geq 0
\end{cases}
\]

(a) Sketch \( H(t) \). Find \( \int_{-2}^{4} H(t) \, dt \).

(b) Sketch \( H(t - 2) \). Find \( \int_{-2}^{4} H(t - 2) \, dt \).

(c) Sketch \( H(t - 2) + H(t) \). Find \( \int_{-2}^{4} [H(t - 2) + H(t)] \, dt \).

**Mixed Review**:

R23.1. Review Chapter 4: 1. (Numerical integration)

R23.2. §6.2: 70. (Graph exponential)

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**HOMEWORK DAY 24 — Fundamental Theorem, Part I**

**Level 1**: §4.3: 3,9,11

**Level 2**:
1. §4.2: 57-61.
2. §4.3: 2,7,10,12,15,17,49
3. Find \( F'(x) \) where \( F(x) = \sin(-x^2) \int_{0}^{x} e^{t^2} \, dt \).
4. Use geometry to evaluate \( \int_{0}^{1} \sqrt{4 - x^2} \, dx \).

**Mixed Review**:

R24.1. A particle moves on a straight line according to the law of motion \( s = t^3 - 9t^2 + 15t + 10, \ t \geq 0 \), where \( t \) is measured in seconds and \( s \) measures the distance in feet of the particle from a fixed point \( O \).

(a) Find the velocity at time \( t \).
(b) When is the particle at rest?
(c) When is the particle moving in the negative direction?
(d) When is the particle moving in the positive direction?
(e) Find the total distance traveled during the first 8 s.
(f) What is the particle's displacement in the first 8 s?

Draw a diagram that illustrates the motion of the particle as in Figure 2, page 165.

R24.2. Chapter 2 Review: 60. (Given graphs of \( f, g \), find derivatives of \( fg, f/g, f \circ g \))

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**HOMEWORK DAY 25 — Fundamental Theorem, Part II**

**Level 1**: §4.3: 19,25,27,32

**Level 2**:
1. §4.3: 5
2. Evaluate the following definite integrals. All of these can be done by inspection. For example, to find \( \int \sec^2(\pi x) \, dx \) you may guess the antiderivative to be \( \tan(\pi x) \), but \( \frac{d}{dx}[\tan(\pi x)] = \pi \sec^2(\pi x) \). So the correct antiderivative is \( \int \sec^2(\pi x) \, dx = \frac{1}{\pi} \tan(\pi x) + C \). For these simple problems, always check the derivative of the antiderivative you found.
(a) \[ \int_1^4 \frac{x - 1}{\sqrt{x}} \, dx \]
(b) \[ \int_0^2 (y - 1)(2y + 1) \, dy \]
(c) \[ \int_0^1 (e^x + e^{-x})^2 \, dx \]
(d) \[ \int_0^1 (x^e + e^x) \, dx \]
(e) \[ \int_1^2 \frac{s^4 + 1}{s^2} \, ds \]
(f) \[ \int_1^3 \sqrt{3x} \, dx \]
(g) \[ \int_0^\pi \sin(2x) \, dx \]
(h) \[ \int_\pi/3^\pi \cos(x/2) \, dx \]
(i) \[ \int_{-2}^2 x^2 \sin(x) \, dx \]

(j) \[ \int_0^2 \frac{dx}{ex^2} \]
(k) \[ \int_0^{\pi/8} \sec^2(2\theta) \, d\theta \]

(l) \[ \int_{-2}^2 f(x) \, dx \text{ where } f(x) = \begin{cases} 
2 & \text{if } -2 \leq x < 0 \\
4 - x^2 & \text{if } 0 < x \leq 2
\end{cases} \]

(m) \[ \int_0^\pi f(x) \, dx \text{ where } f(x) = \begin{cases} 
\cos x & \text{if } 0 \leq x \leq \pi/2 \\
\sin x & \text{if } \pi/2 < x \leq \pi
\end{cases} \]

Mixed Review :
R25.1. Find \( dy/dx \). Simplify answer. (a) \( y = \tan^2 x \)  (b) \( xy^4 + x^2y = x + 3y \)  (c) \( \sin(xy) = x^2 - y^2 \)

R25.2. Find the absolute maximum value of the function \( f(x) = x - e^x \). (include a graph obtained with the information you found)

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**HOMEWORK DAY 26 — Indefinite integrals. Net change.**

**Level 1:** §4.4: 23,25,27,30,42,55

**Level 2:**
§4.4: 29,33,41,47-53,56,59,63

Mixed Review :
R26.1. (a) Explain why \( 1 + u \leq e^u \leq 1 + (e - 1)u \) if \( u \in [0, 1] \)
(b) Explain why \( 1 + \frac{x^2}{4} \leq e^{x^2/4} \leq 1 + (e - 1)\frac{x^2}{4} \) if \( x \in [0, 2] \).
(c) Show that \( 2.66 < \int_0^2 e^{x^2/4} \, dx < 3.15 \)

R26.2. Consider the function \( g(t) = \frac{t^2 - 4}{t + 2} \).
(a) State the domain of \( g \).
(b) Find \( \lim_{t \to -2} g(t) \).
(c) Sketch a graph of the function.
(d) The limit in (b) happens to be the derivative \( f'(a) \) of some function \( f \) at some point \( a \). Find \( f \) and \( a \).

R26.3. Consider the function \( g(h) = \frac{(2 + h)^3 - 8}{h} \).
(a) State the domain of \( g \).
(b) Find \( \lim_{h \to 0} g(h) \).
(c) Sketch a graph of \( g \). Use calculus to find the absolute minimum value of the function and clearly indicate the value in your graph.
(d) The limit in (b) happens to be the derivative \( f'(a) \) for some function \( f \) at some point \( a \). Find \( f(x) \) and \( a \).

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**HOMEWORK DAY 27 — Substitution**

**Level 1:** §4.5: 11,17,29,39,47,71. §6.2: 83
Mixed Review:
R27.1. If \( f \) and \( g \) are differentiable, find

\[
\begin{align*}
(a) \quad & \frac{d}{dx} \left( \sqrt{f(x)} \right) \\
(b) \quad & \frac{d}{dx} \left( f(\sqrt{x}) \right) \\
(c) \quad & \frac{d}{dx} \left( \sqrt{x}f(x) \right) \\
(d) \quad & \frac{d}{dx} \left( f(g(x)) \right) \\
(e) \quad & \frac{d}{dx} \left( f(f(x)) \right) \\
(f) \quad & \frac{d}{dx} \left( \sqrt{f(x)}g(x) \right)
\end{align*}
\]

R27.2. Water is draining into a tank at the rate of \( r(t) = 10\sqrt{t} + 2\pi \sin(\pi t) \) gallons per hour. The initial amount of water in the tank was 300 gallons.

(a) Write an initial value problem for the volume of water in the tank after \( t \) hours.
(b) Write an integral expression for the amount of water in the tank after 9 hours.
(c) Find the amount of water in the tank after 9 hours.

R27.3. §6.2: 23,24,28,29,30 (Limits of exponentials)

**HOMEWORK DAY 28 — Areas between curves**

**Level 1:** §5.1: 1,4,11

**Level 2:**

1. §5.1: 4.
2. In these problems, sketch the region and set up an integral that describes the area. You do not have to evaluate the integral. §5.1: 8,12,18,19.
3. Answer these problems completely. §5.1: 26,31,32.

**Mixed Review:**

R28.1. §4.4: 30

R28.2. Review Chapter 4, 44. (velocity, distance travelled, displacement)

R28.3. Review Chapter 4, 33-38.

**HOMEWORK DAY 29 — Volumes of solids of revolution**

**Level 2:** (no Level 1)

1. Set up an integral for the volume of the following solids. In each case, draw the region in the plane that is revolved about an axis, sketch the volume of the solid of revolution, and highlight one slice of that volume, showing its width and its radius. §5.2: 4,5,7,13,14.
   You do not have to evaluate the integrals, but should know how to if needed.
2. Here as well: draw the given region in the plane. For each subproblem, sketch the volume of the solid of revolution, and highlight one slice of that volume, showing its width and its radius.
   Set up an integral for the volumes of the solids obtained by rotating the region bounded by the curves \( y = x \) and \( y = x^2 \) about the following lines:
   (a) the x-axis (b) the y-axis  (c) \( y = 2 \) (d) \( x = -1 \)
   You do not need to evaluate the integrals, but know how to if asked.

**Mixed Review:**

R29.1. Evaluate \( \int_{0}^{1} x + \sqrt{1 - x^2} \, dx \).

R29.2. Use the properties of integrals to find the following

(a) If \( \int_{2}^{12} g(x) \, dx = -6 \) and \( \int_{2}^{6} g(x) \, dx = -12 \), find \( \int_{6}^{12} g(x) \, dx \)
(b) If \( \int_{0}^{4} (f(u) - u) \, du = 6 \), find \( \int_{0}^{4} f(u) \, du \)
(c) If \( \int_0^4 x \sin(x^3) \, dx \approx 0.359 \), approximately find \( \int_{-4}^4 s \sin(s^3) \, ds \).

R29.3. Chapter 4 Review: 9-18.

R29.4. Are the following equalities true or false? Explain.
(a) \( \int_1^4 \sin(x^2) \, dx = \int_1^4 \sin(u^2) \, du \)  
(b) \( \int_1^2 x \sin(x^2) \, dx = \frac{1}{2} \int_1^4 \sin(x) \, dx \)

R29.5. Chapter 4 Review: 19-28, 49.

**HOMEWORK DAY 30 — Work**

**Level 1:** §5.4: 3.

**Level 2:**
1. §5.4: 5, 7, 15, 19
2. §5.2: 48

**Mixed Review:**
R30.1. A pipe drains and fills a reservoir at a rate \( dV/dt \) indicated by the graph in the figure, where \( V \) is the volume of water in the reservoir. There are initially 50 gallons of water in the reservoir.

(a) When did water flow into the reservoir? When did water flow out of the reservoir? When was there no flow in or out of the reservoir?
(b) How many gallons flowed in through the pipe?
(c) How many gallons flowed out through the pipe?
(d) How many gallons are in the reservoir at time \( t = 12 \) minutes?

R30.2. Review Chapter 4, 4. (Express limit of Riemann sum as a n integral)

**HOMEWORK DAY 33 — Arclength**

**Level 1:** None

**Level 2:**
1. §8.1: 1, 7, 10

**Mixed Review:**
R33.1. Find the derivatives of the following functions. Simplify your answer.
(a) \( g(s) = \int_1^s \frac{t^2}{1 + t^3} \, dt \)  
(b) \( s(t) = \int_t^5 \cos(s^2) \, ds \)  
(c) \( f(x) = \int_t^1 \frac{\sin x}{1 + t^2} \, dt \)

R33.2. (a) Use calculus to find the volume of a sphere of radius \( R \).
(b) Find the volume of the sphere after a hole of radius \( R/2 \) has been drilled through its center.
(c) What percentage of the volume of the whole sphere is left after drilling the hole?
R33.3. Given the equation \( x \sin(\pi x) = \int_0^{x^2} f(t) \, dt \), find \( f(4) \).

R33.4. Review Chapter 4, 8. (Integration and differentiation)

**HOMEWORK DAY 34 — Average value of a function**

**Level 1:** §5.5: 1,2,3,6,7,17.

**Level 2:** due

1. §5.5: 15

2. An aluminum rod is 8cm long. The temperature of the rod at a point \( x \) cm from one end is given by

   \[
   T(x) = \begin{cases} 
   x^3 - x^2 + 32, & 0 \leq x \leq 2 \\
   36 - 2x + x^2, & 2 < x \leq 8 
   \end{cases}
   \]

   Find the average temperature of the rod.

3. §5.5: 18 (avg avg vs maximal blood flow)

4. Consider the function \( f(x) = \sin(x) - \cos(x) \).
   
   (a) Find the average \( f_{\text{av}} \) of \( f \) over the interval \([a, a + \pi]\)?
   
   (b) For what value of \( a \in [0, \pi] \) is the average on \([a, a + \pi]\) largest?

   (c) The figure shows a graph of \( f \). Show the interval \([a, a + \pi]\) over which the average is maximal and the line \( y = f_{\text{av}} \) for that interval. Explain why for any other \( a \) the average would clearly be less.

**Mixed Review :**

R34.1. For the following quantities, write down expressions for the average rate of change over an interval, and for the instantaneous rate of change. In each case, what do these quantities mean? What are their units?

   (a) The number of bees in a beehive \( P(t) \) after \( t \) hours from some starting time. If \( P'(20) = 100 \), what does it mean in practice?

   (b) The surface area \( S(r) \) of a sphere of radius \( r \) cm. If \( S'(10) = 80\pi \), what does it mean in practice?

   (c) Use linearization to approximate the change in surface area from \( r = 10 \) to \( r = 12 \). Compare to the actual change.

R34.2. A population of honeybees increased at a rate of \( r(t) \) bees per week, where the graph of \( r \) is given in problem 47, in the Chapter 4 Review on p 340.

   (a) Use the midpoint rule with \( n = 6 \) intervals to approximate the increase in the bee population during the first 24 weeks.

   (b) If the initial population was 12000 bees, approximately what is the population after the 24 weeks?

R34.3. §4.4: 47.

R34.4. (a) Solve the differential equation \( f'(x) = 8x - 3\sec^2 x \)

   (b) Solve the initial value problem \( f'(u) = \frac{u^2 + \sqrt{u}}{u} \), \( f(1) = 3 \).
Definition:
A function $f(x)$ is continuous at $x = a$ if

$$f(a) = \lim_{x \to a} f(x)$$

or, equivalently, if

$$f(a) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

Definition:
A function $f(x)$ is continuous on a closed interval $[a, b]$ if

1. it is continuous at all points $x \in (a, b)$
2. it is continuous from the right at $x = a$, $f(a) = \lim_{x \to a^+} f(x)$
3. it is continuous from the left at $x = b$, $f(b) = \lim_{x \to b^-} f(x)$

Theorem: All polynomials, rational functions, roots, and trigonometric functions are continuous everywhere in their domain.

Intermediate Value Theorem (IVT):
If a function is continuous on $[a, b]$, then it takes on any value between $f(a)$ and $f(b)$. That is, if $y$ is between $f(a)$ and $f(b)$, there is a $c \in (a, b)$ such that $f(c) = y$.

Extreme Value Theorem (EVT):
If a function is continuous on $[a, b]$, then it takes on a maximum and minimum value in $[a, b]$. That is, there is a $c \in [a, b]$ such that $f_{\text{max}} = f(c) \geq f(x)$ for all $x \in [a, b]$. Similarly with $f_{\text{min}}$.

Level 1: None

Level 2:

1. State all the points where the following functions are continuous.
   (a) $f(x) = \frac{x^2 + 1}{2x^2 - x - 1}$
   (b) $f(x) = \frac{x^2 - 9}{x - 3}$
   (c) $f(x) = \tan(x)$

2. Consider the function

$$f(x) = \begin{cases} 
\cos(x) & , \quad x < 0 \\
2 & , \quad x = 0 \\
1 - x & , \quad 0 < x \leq 2 \\
\frac{1}{2}(x - 2)^2 - 1 & , \quad 2 < x \leq 4 \\
2 & , \quad x > 4 
\end{cases}$$

   (a) Sketch the graph of the function.
   (b) State all values of $a$ for which $\lim_{x \to a} f(x)$ exists. Why does the limit exist at these points?
   (c) State all values of $a$ for which the function is continuous. Why is it not continuous at the remaining points?

3. Consider the function

$$f(x) = \begin{cases} 
 cx + 2x & , \quad x < 2 \\
 x^3 - cx & , \quad x \geq 2 
\end{cases}$$

   (a) For what value of the constant $c$ is $f$ continuous on $(-\infty, \infty)$?
   (b) Sketch the graph of the function if $c$ has the value you found in (a).
4. Suppose that anyone with a taxable income of \( x \) owes 10\% of his income in taxes if \( x \leq 30,000 \) and 10\% of $30,000 plus 25\% of all his income above $30,000, if \( x > 30,000 \).
(a) Find the taxes that two people A and B owe, if A makes $20,000 and B makes $45,000.
(b) Write down a function \( T(x) \) that returns the tax owed by someone with income \( x \).
(c) Sketch a graph of \( T(x) \).
(d) Is the function continuous? Explain (mathematically) why.

5. (a) If a function is continuous on \([a,b]\), what can you conclude, regarding absolute extrema of the function?
(b) How do you find the absolute extrema of a function that is continuous on a closed interval?
(c) Find the absolute maxima and minima of \( f(t) = 2 \cos t + \sin(2t) \) over the interval \([0, \pi/2]\).
(Find the answer without using a calculator.)

6. Prove that the equation \( \cos x = x \) has a root in \([0, \pi/2]\). (Hint: Consider the function \( f(x) = \cos x - x \) and use the IVT.)

Mixed Review:
R35a.1. A particle moves on a vertical line so that its coordinate at time \( t \) is \( y = t^3 - 12t + 3, t \geq 0 \).
(a) Find the distance that the particle travels in the time interval \( 0 \leq t \leq 3 \).
(b) Find the particle’s displacement in the time interval \( 0 \leq t \leq 3 \).
R35a.2. Chapter 4 Concept Check, 5.
R35a.3. Find the local and absolute extreme values of the function on the given interval
\[ f(x) = 10 + 27x - x^3, \quad [0, 4] \]
R35a.4. Short answer questions.
(a) A honeybee population starts with 100 bees and increases at a rate on \( n'(t) \) bees per week. What does
\[ 100 + \int_0^{15} n'(t) \, dt \]
represent?
(b) If the units of \( x \) are feet and the units for \( a(x) \) are pounds per foot, what are the units for \( da/dx \)? what units does \( \int_2^8 a(x) \, dx \) have?

HOMEWORK DAY 35b — Differentiability. MVT.

\begin{definition}
A function \( f(x) \) is \textit{differentiable at} \( x = a \) if
\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]
exists, or, equivalently, if
\[ \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} . \]
If the function is continuous, differentiable away from \( a \), and the limits \( \lim_{x \to a^+} f'(x) \) and \( \lim_{x \to a^-} f'(x) \) exist, then an equivalent condition for differentiability is that
\[ \lim_{x \to a^+} f'(x) = \lim_{x \to a^-} f'(x) \]
In each case, the limit is the derivative at \( x = a \), \( f'(a) \).
Theorem: If a function is differentiable at a point \( x = a \), then it must be continuous there.

Mean Value Theorem (MVT):
If a function is continuous in \([a, b]\) and differentiable in \((a, b)\), then there exists a \( c \in (a, b) \) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Level 1: §3.2: 9.

Level 2:

1. (a) Give an example of a function that is continuous but not differentiable at a point, if possible.
   (b) Give an example of a function that is differentiable but not continuous at a point, if possible.

2. Give a formula for \( g' \) and sketch the graphs of \( g \) and \( g' \).
   (a) \( g(x) = |x^2 + x| \)
   (b) The function \( g \) given in page 139, # 94.

3. The gravitational force exerted by the earth on a unit mass at a distance \( r \) from the center of the planet is
\[
F(r) = \begin{cases} 
  \frac{GMr}{R^3} & \text{if } 0 \leq r < R \\
  \frac{GM}{r^2} & \text{if } r \geq R
\end{cases}
\]
   where \( M \) is the mass of the earth, \( R \) is its radius, and \( G \) is the gravitational constant (all three are positive constants).
   (a) Where is \( F \) continuous? Why?
   (b) Where is \( F \) differentiable? Explain. Find a formula for \( F'(r) \).
   (c) Sketch a clearly labelled graph of \( F(r) \).

4. Consider the function
\[
f(x) = \begin{cases} 
  cx + 2x & , \ x < 2 \\
  x^3 - cx & , \ x \geq 2
\end{cases}
\]
   (a) For what value of the constant \( c \) is \( f \) continuous on \((-\infty, \infty)\)?
   (b) If \( c \) has the value you found in (a), is \( f \) differentiable at \( x = 2 \)? Why or why not? Write down a formula for \( f'(x) \).
   (b) If \( c \) does not have the value you found in (a), is \( f \) differentiable at \( x = 2 \)? Why or why not?

5. State the Mean Value Theorem. Include a picture and explain its meaning geometrically.

6. §3.2: 13,23,24 (explain your answer!)

Mixed Review:

R35b.1. Find a linear approximation \( L(x) \) of \( f(x) = \sqrt{1-x} \) at \( x = 0 \) and use it to approximate the numbers \( \sqrt{0.9} \) and \( \sqrt{0.99} \). Sketch a graph of both \( f \) and \( L \).

R35b.2. Use linear approximations to estimate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemispherical dome with diameter 50 m.

R35b.3. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Approximate the area of the pool by
   (a) left endpoint rule
   (b) midpoint rule (using half as many intervals)
R35b.4. Consider a cylindrical shell of inner radius $r$, height $h$, and thickness $\Delta r$.
(a) Use linearization to approximate the volume of the shell.
(b) Find the exact volume and compare to the approximation in (b).

R35b.5. If $f$ is continuous on $[0,1]$, prove that $\int_0^1 f(x) \, dx = \int_0^1 f(1-x) \, dx$.

**HOMEWORK DAY 36 — Integrability. More applications (piecewise acceleration).**

**Definition:** A function $f(x)$ is Riemann integrable on $[a, b]$ if the following limit exists,
\[
\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x^*_k)\Delta x
\]
where $x^*_k$ is any point in the kth subinterval $[x_{k-1}, x_k]$ of length $\Delta x$.

**Theorem:** If a function is continuous on $[a,b]$ then it is integrable over $[a,b]$. Same is true if piecewise continuous with finite jump discontinuities.

Sample problems of interest:

**Example:** A car accelerates from rest for 5 seconds with acceleration of $3 \text{ m/sec}^2$. It then continues with zero acceleration. Find the total distance travelled in the first 2 minutes. (Write down the initial value problem that governs the position of the car in the first 5 seconds. Write down the initial value problem that governs the position in the next 1:55 min. Sketch a graph of acceleration, velocity and position travelled.)

**Example:** The engineer of a freight train needs to stop in 9250 ft to avoid striking a barrier. The train is traveling at a speed of 60 mi/hr. The engineer applies one set of brakes, which causes him to decelerate at a rate of $0.2 \text{ mi/min}^2$ for 15 s. He realizes he is not going to make it, so he applies a second set of brakes, which causes the train to decelerate at a rate of $0.3 \text{ mi/min}^2$. Will he strike the barrier?

**Level 1:** None

**Level 2:**
1. A model rocket is fired vertically upward from rest. Its acceleration for the first three seconds is $a(t) = 60t$ (in ft/sec$^2$), at which time the fuel is exhausted and it becomes a freely “falling” body.
   (a) Sketch a graph of the acceleration as a function of time, for $t \geq 0$.
   Sketch a graph of the rocket velocity vs time, for $t \geq 0$.
   Sketch a graph of the height of the rocket above ground vs time, for $t \geq 0$.
   Note that the velocity and position are continuous functions of time.
   (b) Write down the initial value problem that governs the motion over the first 3 seconds. Solve it. Where is the rocket at $t = 3$ sec, and what is its velocity?
   (c) Write down the initial value problem that governs the motion for $t > 3$ sec. Solve it. When does the rocket hit the ground? What is the rocket’s velocity when it hits the ground?
2. The derivative of a function is given by the formula
   \[
f'(x) = \begin{cases} 
2 & \text{if } 0 < x < 2 \\
-1 & \text{if } 2 < x < 3 \\
0 & \text{if } x > 3
\end{cases}
\]
   (a) Sketch the graph of $f'(x)$. 

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(b) Sketch the graph of \( f(x) \), if \( f(0) = -1 \) and \( f \) is continuous on \( [0, \infty) \).

(c) Find a formula for \( f(x) \).

3. §3.9: 69

4. §3.9: 70

Mixed Review :

R36.1. Show that \( \sqrt{1 + 2x} \approx 1 + \frac{x}{2} \) if \( x \approx 0 \). Use this result to approximate \( \sqrt{0.9} \).

R36.2. §2.9, #38 (flux of blood through vessel: Poiseuille’s Law)

R36.3. Average value of a function defined piecewise.

(a) Find a function \( f \) such that \( F(x) = \int_0^x f(t) \, dt \) for the function \( F \) whose graph is shown in the figure at right.

(b) What is the average of \( f \) over \([0,3]\)?

(c) What is the average of \( F \) over \([0,3]\)?

R36.4. §4.5: 44,46,58

R36.5. Answer the following questions.

(a) What is the difference between \( \int f(x) \, dx \), \( \int_a^x f(t) \, dt \), and \( \int_a^b f(t) \, dt \).

(b) What is the difference between \( \frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] \) and \( \frac{d}{dx} \left[ \int_a^b f(t) \, dt \right] \), where \( a \neq b \), if any?