

A summary of different types of equilibrium solution

$$\bar{\dot{y}} = P \bar{y} \implies \text{equilibrium solution } \bar{y}_E = \bar{0} \text{ (the origin)}$$

① eigenvalues of P are real & distinct

- a) $r_1 & r_2$ have opposite signs \Rightarrow origin is an asymptotically unstable saddle point
- b) $r_1, r_2 < 0$ \Rightarrow origin is an asymptotically stable node
- c) $r_1, r_2 > 0$ \Rightarrow origin is an asymptotically unstable node

② Some eigenvalues are complex conjugate

- a) $\text{real}(r_1) = \text{real}(r_2) < 0$ \Rightarrow origin is an asymptotically stable spiral point
- b) $\text{real}(r_1) = \text{real}(r_2) > 0$ \Rightarrow origin is an asymptotically unstable spiral point
- c) $\text{real}(r_1) = \text{real}(r_2) = 0$ \Rightarrow origin is a stable center
(neither asymptotically stable nor asymptotically unstable)

③ some eigenvalues (either real or complex) are repeated

- a) $r_1 = r_2 > 0$ \Rightarrow origin is an asymptotically unstable improper node
- b) $r_1 = r_2 < 0$ \Rightarrow origin is an asymptotically stable improper node

System 1 $\bar{y}' = P \bar{y} , \quad P = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

1) ansatz: $\bar{y} = \bar{\xi} e^{rt} \xrightarrow{\text{ODE}} P\bar{\xi} = r\bar{\xi} \Rightarrow (P - rI)\bar{\xi} = \bar{0}$

2) eigenvalues of P : $\det(P - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0 \Rightarrow (1-r)^2 - 4 = 0$
 $\Rightarrow 1-r = \pm 2 \Rightarrow \begin{cases} r_1 = 3 \\ r_2 = -1 \end{cases}$

3) eigenvectors of P : $(P - rI)\bar{\xi} = \bar{0}$

$$r_1 = 3 \Rightarrow \begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -2\xi_1 + \xi_2 = 0 \\ 4\xi_1 - 2\xi_2 = 0 \end{cases} \Rightarrow \xi_2 = 2\xi_1 \Rightarrow \boxed{\begin{array}{l} (1) \\ \bar{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{array}}$$

$$r_2 = -1 \Rightarrow \begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2\xi_1 + \xi_2 = 0 \\ 4\xi_1 + 2\xi_2 = 0 \end{cases} \Rightarrow \xi_2 = -2\xi_1 \Rightarrow \boxed{\begin{array}{l} (2) \\ \bar{\xi} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{array}}$$

4) we obtain two solution families: $\bar{y}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$ and $\bar{y}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

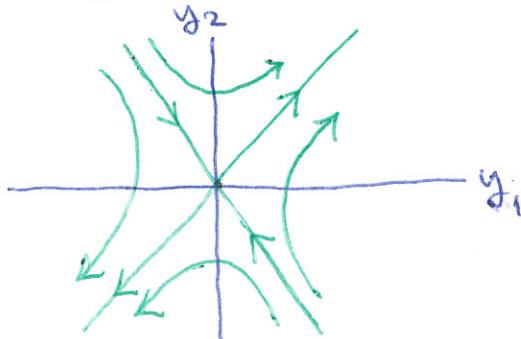
5) Verify that the two solution families $\bar{y}^{(1)}$ and $\bar{y}^{(2)}$ form a fundamental set:

$$W[\bar{y}^{(1)}, \bar{y}^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -2e^{2t} - 2e^{2t} = -4e^{2t} \neq 0. \quad \underline{\text{O.K.}}$$

6) general solution: $\bar{y}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

7) see Matlab code ch7_system1.m for the plot of phase portrait.

since $r_1 > 0$ and $r_2 < 0$, the origin (which is the equilibrium solution) is an asymptotically unstable saddle point.



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System 2 $\bar{y}' = P \bar{y}$, $P = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}$, $\bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1) ansatz: $\bar{y} = \bar{\xi} e^{rt} \xrightarrow{\text{ODE}} P\bar{\xi} = r\bar{\xi} \Rightarrow (P - rI)\bar{\xi} = \bar{0}$

2) eigenvalues of P : $\det(P - rI) = 0 \Rightarrow \begin{vmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{vmatrix} = 0 \Rightarrow (-\frac{1}{2} - r)^2 + 1 = 0$
 $\Rightarrow -\frac{1}{2} - r = \pm i \Rightarrow \begin{cases} r_1 = -\frac{1}{2} + i \\ r_2 = -\frac{1}{2} - i \end{cases}$ eigenvalues are complex conjugate.

3) eigenvectors of P : $(P - rI)\bar{\xi} = \bar{0}$

$$r_1 = -\frac{1}{2} + i \Rightarrow \begin{bmatrix} -\frac{1}{2} + \frac{1}{2} - i & 1 \\ -1 & -\frac{1}{2} + \frac{1}{2} - i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -i\xi_1 + \xi_2 = 0 \\ -\xi_1 - i\xi_2 = 0 \end{cases} \Rightarrow i\xi_1 = \xi_2 \Rightarrow \begin{cases} \xi_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \\ \xi_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{cases}$$

$$r_2 = -\frac{1}{2} - i \Rightarrow \dots \dots \dots \dots \dots \dots \dots \dots \Rightarrow \begin{cases} \xi_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \\ \xi_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{cases}$$

NOTE: when eigenvalues are complex conjugate, eigenvectors will also be conjugate

4) We obtain two (complex-valued) solution families:

$$\bar{y}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-\frac{1}{2}+i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{t}{2}} e^{it} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{t}{2}} (\cos t + i \sin t) = \begin{pmatrix} e^{-\frac{t}{2}} \cos t + i e^{-\frac{t}{2}} \sin t \\ i e^{-\frac{t}{2}} \cos t - e^{-\frac{t}{2}} \sin t \end{pmatrix}$$

$$\Rightarrow \bar{y}^{(1)} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{t}{2}} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{t}{2}}$$

$$\bar{y}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-\frac{1}{2}-i)t} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-\frac{t}{2}} e^{-it} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-\frac{t}{2}} (\cos t - i \sin t) = \begin{pmatrix} e^{-\frac{t}{2}} \cos t - i e^{-\frac{t}{2}} \sin t \\ -i e^{-\frac{t}{2}} \cos t - e^{-\frac{t}{2}} \sin t \end{pmatrix}$$

$$\Rightarrow \bar{y}^{(2)} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{t}{2}} - i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{t}{2}}$$

if $\begin{cases} \bar{u} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{t}{2}} \\ \bar{v} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{t}{2}} \end{cases} \Rightarrow$ we observe that $\begin{cases} \bar{y}^{(1)} = \bar{u} + i \bar{v} \\ \bar{y}^{(2)} = \bar{u} - i \bar{v} \end{cases}$

We therefore choose \bar{u} and \bar{v} as two (real-valued) solution families.

5) verify that the two solution families \bar{u} and \bar{v} form a fundamental set:

$$W[\bar{u}, \bar{v}](t) = \begin{vmatrix} \cos t \cdot \bar{e}^{-\frac{t}{2}} & \sin t \cdot \bar{e}^{-\frac{t}{2}} \\ -\sin t \cdot \bar{e}^{-\frac{t}{2}} & \cos t \cdot \bar{e}^{-\frac{t}{2}} \end{vmatrix} = (\cos^2 t + \sin^2 t) \bar{e}^{-\frac{t}{2}} = \bar{e}^{-\frac{t}{2}} \neq 0. \quad \underline{\text{O.K.}}$$

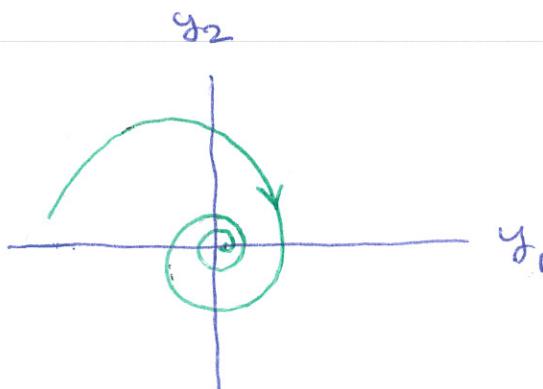
6) general solution: $\bar{y}(t) = c_1 e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

7) IC: $\bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$

unique solution: $y(t) = e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \Rightarrow \boxed{y(t) = \begin{pmatrix} \cos t + \sin t \\ \cos t - \sin t \end{pmatrix} e^{-t/2}}$

8) See Matlab code ch_system2.m for the plots of unique solution and the phase portrait.

Since the real part of eigenvalues is negative, the origin (which is the equilibrium solution) is an asymptotically stable spiral point.



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System 3 $\bar{y}' = P \bar{y}$, $P = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, IC: $\bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1) ansatz: $\bar{y} = \bar{\xi} e^{rt}$ $\xrightarrow{\text{ODE}}$ $P \bar{\xi} = r \bar{\xi} \Rightarrow (P - rI) \bar{\xi} = \bar{0}$

2) eigenvalues of P: $\det(P - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = 0 \Rightarrow (1-r)(3-r) + 1 = 0$
 $\Rightarrow r^2 - 4r + 4 = 0 \Rightarrow r_1 = r_2 = 2$ repeated eigenvalues

3) eigenvectors of P: since there is a double eigenvalue, there will be only one eigenvector:

$$(P - rI) \bar{\xi} = \bar{0} \Rightarrow \begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -\xi_1 - \xi_2 = 0 \\ \xi_1 + \xi_2 = 0 \end{cases} \Rightarrow \xi_2 = -\xi_1 \Rightarrow \bar{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

this means we only have one solution family: $\bar{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$.

4) we need to find a second solution family $\bar{y}^{(2)}$.

we have seen in class that a proper choice is $\bar{y}^{(2)} = \bar{\gamma} t e^{2t} + \bar{\eta} e^{2t}$
and we will try to find $\bar{\gamma}$ and $\bar{\eta}$.

Let $\bar{y} = \bar{\gamma} t e^{2t} + \bar{\eta} e^{2t}$. $\Rightarrow \bar{y}' = \bar{\gamma} e^{2t} + 2\bar{\gamma} t e^{2t} + 2\bar{\eta} e^{2t}$

$$\underline{\bar{y}' = P \bar{y}} \quad \underline{\bar{\gamma} e^{2t}} + 2\underline{\bar{\gamma} t e^{2t}} + 2\underline{\bar{\eta} e^{2t}} - \underline{P \bar{\gamma} t e^{2t}} - \underline{P \bar{\eta} e^{2t}} = \bar{0}$$

$$\Rightarrow (2\bar{\gamma} - P\bar{\gamma}) t e^{2t} + (\bar{\gamma} + 2\bar{\eta} - P\bar{\eta}) e^{2t} = \bar{0}$$

$$\Rightarrow \begin{cases} 2\bar{\gamma} - P\bar{\gamma} = \bar{0} \\ \bar{\gamma} + 2\bar{\eta} - P\bar{\eta} = \bar{0} \end{cases} \Rightarrow \bar{\gamma} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ which is the eigenvector } \bar{\xi}.$$

$$\bar{\gamma} + 2\bar{\eta} - P\bar{\eta} = \bar{0} \Rightarrow \bar{\gamma} = (P - 2I)\bar{\eta} \Rightarrow \text{we find } \bar{\eta} \text{ as follows.}$$

$$(P - 2I)\bar{\eta} = \bar{\gamma} \Rightarrow \begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} -\eta_1 - \eta_2 = 1 \\ \eta_1 + \eta_2 = -1 \end{cases} \Rightarrow \eta_2 = -\eta_1 - 1$$

$$\Rightarrow \bar{\eta} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \bar{y}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} \Rightarrow \bar{y}^{(2)} = \begin{pmatrix} t+1 \\ -t-2 \end{pmatrix} e^{2t}$$

5) verify that the two solution families $\bar{y}^{(1)}$ and $\bar{y}^{(2)}$ form a fundamental set:

$$W\left[\bar{y}^{(1)}, \bar{y}^{(2)}\right](t) = \begin{vmatrix} e^{2t} & (t+1)e^{2t} \\ -e^{-2t} & (-t-2)e^{2t} \end{vmatrix} = (-t-2 + t+1) e^{4t} = -e^{4t} \neq 0 \text{ - O.K.}$$

6) general solution: $\bar{y}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t+1 \\ -(t+2) \end{pmatrix} e^{2t}$

7) IC: $\bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -c_1 - 2c_2 \end{pmatrix}$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ -c_1 - 2c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -2 \end{cases}$$

unique solution: $y(t) = \begin{pmatrix} 3 \\ -3 \end{pmatrix} e^{2t} + \begin{pmatrix} -2t-2 \\ 2t+4 \end{pmatrix} e^{2t} \Rightarrow \boxed{y(t) = \begin{pmatrix} 1-2t \\ 1+2t \end{pmatrix} e^{2t}}$

8) See Matlab code ch_system3.m for the plots of unique solution and the phase portrait.

Since $r_1 = r_2 > 0$, the origin (which is the equilibrium solution) is an asymptotically unstable improper node.

