

System 4

Use the method of variation of parameters to

find the general solution of

$$\bar{y}' = P \bar{y} + \bar{g}(t) \quad \text{with} \quad P = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \bar{g}(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

Assume $t_0 = 0$.

Moreover, if IC: $\bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find the unique solution.

STEP 1 Find general solution to $\bar{y}' = P \bar{y}$

$$\text{Ansatz: } \bar{y}(t) = \bar{\gamma}(t) e^{rt} \Rightarrow \det(P - rI) = 0$$

$$\Rightarrow \begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = 0$$

$$\Rightarrow (2+r)^2 - 1 = 0 \Rightarrow (2+r-1)(2+r+1) = 0$$

$$\Rightarrow \begin{cases} r_1 = -3 \\ r_2 = -1 \end{cases}$$

To find eigenvectors:

$$r_1 = -3: \quad (P - r_1 I) \bar{\gamma}^{(1)} = \bar{0} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{\gamma}_1 \\ \bar{\gamma}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \bar{\gamma}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$r_2 = -1: \quad (P - r_2 I) \bar{\gamma}^{(2)} = \bar{0} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{\gamma}_1 \\ \bar{\gamma}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \bar{\gamma}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We then obtain two solution families:

$$\begin{cases} \bar{y}^{(1)}(t) = \bar{\gamma}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} = \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix} \\ \bar{y}^{(2)}(t) = \bar{\gamma}^{(2)} e^{r_2 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \end{cases} \Rightarrow Y = \begin{bmatrix} \bar{y}^{(1)} & \bar{y}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

$$W[\bar{y}^{(1)}, \bar{y}^{(2)}](t) = \det(Y) = 2e^{-4t} \neq 0 \Rightarrow \bar{y}^{(1)}(t) \text{ and } \bar{y}^{(2)}(t) \text{ are fundamental solutions}$$

$$\Rightarrow \text{general solution to homogeneous problem: } \bar{y}_c(t) = c_1 \bar{y}^{(1)}(t) + c_2 \bar{y}^{(2)}(t) = Y \bar{c}$$

where $\bar{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

STEP 2 Find a particular solution to $\bar{y}' = P\bar{y} + g(t)$

$$\text{let } \bar{y}_p(t) = Y(t) \bar{u}(t) \Rightarrow \bar{y}'_p(t) = Y' \bar{u} + Y \bar{u}'$$

$$\{\text{ODE}\} \Rightarrow Y' \bar{u} + Y \bar{u}' = PY \bar{u} + \bar{g}$$

$$\{Y' = PY\} \Rightarrow \boxed{Y \bar{u}' = \bar{g}}$$

we will solve this system to find \bar{u}' :

$$\begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \Rightarrow \begin{cases} e^{-3t} u'_1 + e^{-t} u'_2 = 2e^{-t} \\ -e^{-3t} u'_1 + e^{-t} u'_2 = 3t \end{cases}$$

$$\text{We add the two equations} \Rightarrow 2e^{-t} u'_2 = 2e^{-t} + 3t$$

$$\Rightarrow \boxed{u'_2(t) = 1 + \frac{3}{2}t e^t}$$

then we use the first equation to find u'_1 :

$$e^{-3t} u'_1 + e^{-t} \left(1 + \frac{3}{2}t e^t \right) = 2e^{-t} \Rightarrow e^{-3t} u'_1 = e^{-t} - \frac{3}{2}t$$

$$\Rightarrow \boxed{u'_1(t) = e^{-t} - \frac{3}{2}t e^{3t}}$$

We now integrate to find $u_1(t)$ and $u_2(t)$:

$$u_1(t) = \int_{s=0}^{s=t} \left(e^{2s} - \frac{3}{2}s e^{3s} \right) ds = \left[\frac{1}{2} e^{2s} \right]_{s=0}^{s=t} - \frac{3}{2} \left[\frac{(3s-1)}{9} e^{3s} \right]_{s=0}^{s=t}$$

$$\Rightarrow \boxed{u_1(t) = \frac{1}{2} e^{2t} - \frac{3t-1}{6} e^{3t} - \frac{2}{3}}$$

$$u_2(t) = \int_{s=0}^{s=t} \left(1 + \frac{3}{2}s e^s \right) ds = \left[s + \frac{3}{2}(s-1) e^s \right]_{s=0}^{s=t} \Rightarrow \boxed{u_2(t) = t + \frac{3}{2}(t-1) e^t + \frac{3}{2}}$$

general solution: $\bar{y}(t) = \bar{y}_c(t) + \bar{y}_p(t) = c_1 \bar{y}^{(1)}(t) + c_2 \bar{y}^{(2)}(t) + u_1(t) \bar{y}^{(1)}(t) + u_2(t) \bar{y}^{(2)}(t)$

$$\bar{y}(t) = c_1 \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + \left[\frac{1}{2} e^{-t} - \frac{3t-1}{6} e^{-3t} - \frac{2}{3} \right] \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix} + \left[t + \frac{3}{2}(t-1) e^t + \frac{3}{2} \right] \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

It is left to find the unique solution using the IC: $\bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} c_1 + c_2 + \frac{1}{2} + \frac{1}{6} - \frac{2}{3} - \frac{3}{2} + \frac{3}{2} \\ -c_1 + c_2 - \frac{1}{2} - \frac{1}{6} + \frac{2}{3} - \frac{3}{2} + \frac{3}{2} \end{bmatrix} \Rightarrow \begin{cases} 1 = c_1 + c_2 \\ 1 = -c_1 + c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 1 \end{cases}$$

unique solution:

$$\boxed{\bar{y}(t) = \left[\frac{1}{2} e^{2t} - \frac{3t-1}{6} e^{3t} - \frac{2}{3} \right] \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix} + \left[1+t + \frac{3}{2}(t-1) e^t + \frac{3}{2} \right] \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}}$$

In order to plot $y_1(t)$ and $y_2(t)$ versus t , it is easier if we find $y_1(t)$ and $y_2(t)$ separately:

$$\begin{cases} y_1(t) = \frac{1}{2} e^{2t} e^{-3t} - \frac{3t-1}{6} e^{3t} e^{-3t} - \frac{2}{3} e^{-3t} + e^{-t} + t e^{-t} + \frac{3}{2}(t-1) e^t e^{-t} + \frac{3}{2} e^{-t} \\ y_2(t) = -\frac{1}{2} e^{2t} e^{-3t} + \frac{3t-1}{6} e^{3t} e^{-3t} + \frac{2}{3} e^{-3t} + e^{-t} + t e^{-t} + \frac{3}{2}(t-1) e^t e^{-t} + \frac{3}{2} e^{-t} \end{cases}$$

$$\begin{cases} y_1(t) = \frac{1}{2} e^{-t} - \frac{t}{2} + \frac{1}{6} - \frac{2}{3} e^{-3t} + e^{-t} + t e^{-t} + \frac{3}{2} t - \frac{3}{2} + \frac{3}{2} e^{-t} \\ y_2(t) = -\frac{1}{2} e^{-t} + \frac{t}{2} - \frac{1}{6} + \frac{2}{3} e^{-3t} + e^{-t} + t e^{-t} + \frac{3}{2} t - \frac{3}{2} + \frac{3}{2} e^{-t} \end{cases}$$

$$\boxed{\begin{aligned} y_1(t) &= (3+t) e^{-t} - \frac{2}{3} e^{-3t} + t - \frac{4}{3} \\ y_2(t) &= (2+t) e^{-t} + \frac{2}{3} e^{-3t} + 2t - \frac{5}{3} \end{aligned}}$$

See the MATLAB code "Ch7_system4.m" on the course web page.