

Solutions to HW #5

$$\bar{y}' = P \bar{y} + \bar{g}(t), \quad P = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}, \quad \bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

① Find the general solution to $\bar{y}' = P \bar{y}$:

Ansatz: $\bar{y}(t) = \bar{\gamma} e^{rt} \xrightarrow{\text{ODE}} P\bar{\gamma} = r\bar{\gamma}$

$$\det(P - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & -2 \\ 2 & -3-r \end{vmatrix} = (1-r)(-3-r) + 4 = 0$$

$$\Rightarrow r^2 + 2r + 1 = 0 \Rightarrow \boxed{r_1 = r_2 = -1} \rightarrow \boxed{\text{origin is asymptotically stable improper node}}$$

we can find one eigenvalue:

$$(P - rI)\bar{\gamma} = \bar{0} \Rightarrow \begin{pmatrix} 1 - (-1) & -2 \\ 2 & -3 - (-1) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2\gamma_1 - 2\gamma_2 = 0 \\ 2\gamma_1 - 2\gamma_2 = 0 \end{cases}$$

both equ's are the same

$$\Rightarrow 2\gamma_1 - 2\gamma_2 = 0 \Rightarrow \begin{cases} \gamma_1 = 1 \\ \gamma_2 = 1 \end{cases} \Rightarrow \boxed{\bar{\gamma} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\Rightarrow \text{The first solution family is } \boxed{\bar{y}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}}$$

To find the second solution family, we consider the ansatz:

$$\bar{y}^{(2)} = \bar{\gamma} e^{-t} + \bar{n} \bar{e}^{-t} \Rightarrow \bar{y}' = \bar{\gamma} e^{-t} - \bar{\gamma} t e^{-t} - \bar{n} \bar{e}^{-t}$$

$$\xrightarrow{\text{ODE}} \bar{\gamma} e^{-t} - \bar{\gamma} t e^{-t} - \bar{n} \bar{e}^{-t} = P(\bar{\gamma} e^{-t} + \bar{n} \bar{e}^{-t})$$

$$\Rightarrow \underline{(\bar{\gamma} - \bar{n}) \bar{e}^{-t}} - \underline{\bar{\gamma} t \bar{e}^{-t}} = \underline{P \bar{\gamma} e^{-t}} + \underline{P \bar{n} \bar{e}^{-t}} \Rightarrow \begin{cases} P\bar{\gamma} = -\bar{\gamma} \\ P\bar{n} = \bar{\gamma} - \bar{n} \end{cases}$$

$$\begin{cases} P\bar{Y} = -\bar{Y} \\ P\bar{n} = \bar{Y} - \bar{n} \end{cases} \Rightarrow \begin{cases} \boxed{\bar{Y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \\ (P+I)\bar{n} = \bar{Y} \Rightarrow \begin{pmatrix} 1+1 & -2 \\ 2 & -3+1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} 2n_1 - 2n_2 = 1 \\ 2n_1 - 2n_2 = 1 \end{cases} \end{cases}$$

both equ's are the same

$$\Rightarrow 2n_1 - 2n_2 = 1 \Rightarrow \begin{cases} n_1 = 1 \\ n_2 = \frac{1}{2} \end{cases} \Rightarrow \boxed{\bar{n} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}}$$

$$\Rightarrow \boxed{\bar{y}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t} + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{-t}}$$

So far we have obtained two solution families:

$$\bar{y}^{(1)} = \begin{pmatrix} e^{-t} \\ \bar{e}^{-t} \end{pmatrix} \quad \text{and} \quad \bar{y}^{(2)} = \begin{pmatrix} (t+1) \bar{e}^{-t} \\ (t+\frac{1}{2}) \bar{e}^{-t} \end{pmatrix}$$

We check if their wronskian is non-zero:

$$W = \begin{vmatrix} \bar{e}^{-t} & (t+1) \bar{e}^{-t} \\ \bar{e}^{-t} & (t+\frac{1}{2}) \bar{e}^{-t} \end{vmatrix} = (t+\frac{1}{2}) \bar{e}^{-2t} - (t+1) \bar{e}^{-2t} = -\frac{1}{2} \bar{e}^{-2t} \neq 0$$

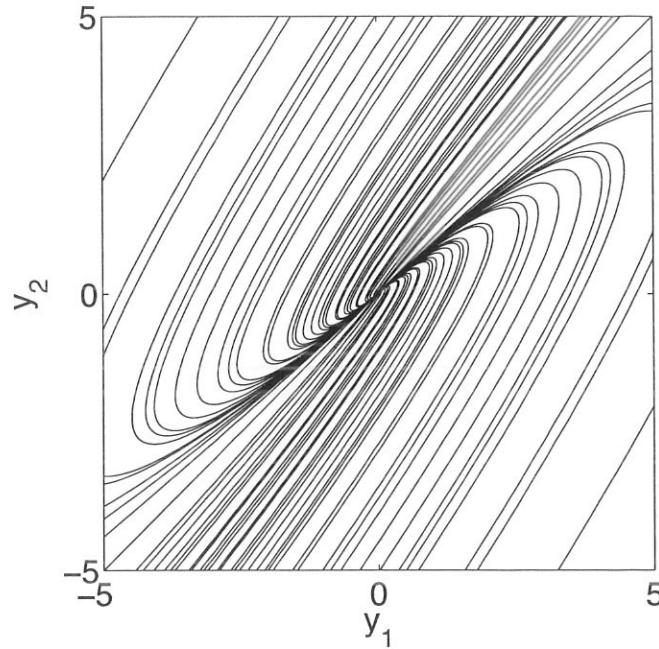
Therefore, $\bar{y}^{(1)}$ and $\bar{y}^{(2)}$ form a fundamental set of solutions and the general solution to the homogeneous ODE system reads:

$$\boxed{\bar{y}_c(t) = c_1 \begin{pmatrix} \bar{e}^{-t} \\ \bar{e}^{-t} \end{pmatrix} + c_2 \begin{pmatrix} (t+1) \bar{e}^{-t} \\ (t+\frac{1}{2}) \bar{e}^{-t} \end{pmatrix}} \rightarrow \text{this is the complementary sol. to the non-homogeneous system.}$$

It can also be written in the matrix form:

$$\bar{y}_c(t) = Y(t) \cdot \bar{c} \quad \text{where} \quad Y(t) = \begin{pmatrix} \bar{e}^{-t} & (t+1) \bar{e}^{-t} \\ \bar{e}^{-t} & (t+\frac{1}{2}) \bar{e}^{-t} \end{pmatrix} \quad \text{and} \quad \bar{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

2. The origin is an asymptotically stable improper node.



This is the Matlab code that generates the above phase portrait.

```
t=linspace(-10,10,1000);
y1=[exp(-t);exp(-t)];
y2=[(t+1).*exp(-t);(t+0.5).*exp(-t)];

figure
set(gca,'fontsize', 20);
hold on

C1=(-2:.5:2); C2=(-2:.5:2);
for i=1:length(C1)
    for j=1:length(C2)
        c1=C1(i); c2=C2(j);
        y=c1*y1+c2*y2;
        plot(y(1,:),y(2,:),'k-')
    end
end
xlabel('y_1')
ylabel('y_2')
axis equal
axis([-5 5 -5 5])
```

- ③ Find a particular sol to $\bar{y}' = P\bar{y} + \bar{g}$ using the method of variation of parameters.

We consider the ansatz for a particular solution: $\bar{y}_p = Y(t) \bar{u}(t)$

$$\Rightarrow \bar{y}'_p = Y' \bar{u} + Y \bar{u}' \xrightarrow{\text{non-homogeneous ODE}} Y' \bar{u} + Y \bar{u}' = P Y \bar{u} + \bar{g}$$

$\left\{ \text{since } Y' = PY \right\} \Rightarrow \boxed{Y \bar{u}' = \bar{g}}$ → we need to solve this linear system to find \bar{u}' .

$$\begin{bmatrix} \bar{e}^t & (t+1)\bar{e}^t \\ \bar{e}^t & (t+\frac{1}{2})\bar{e}^t \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} \bar{e}^t \\ 0 \end{bmatrix}$$

$$\left\{ \bar{e}^t u'_1 + (t+1)\bar{e}^t u'_2 = \bar{e}^t \right.$$

$$\left. \bar{e}^t u'_1 + (t+\frac{1}{2})\bar{e}^t u'_2 = 0 \right.$$

$$\text{Subtract: } \frac{1}{2}\bar{e}^t u'_2 = \bar{e}^t \Rightarrow \boxed{u'_2 = 2}$$

$$\Rightarrow u'_1 = 1 - 2(t+1) \Rightarrow$$

$$\boxed{u'_1 = -1 - 2t}$$

we now integrate to find u_1 and u_2 :

$$u_1(t) = \int_{s=0}^{s=t} (-1 - 2s) ds = \left[-s - s^2 \right]_{s=0}^{s=t} = \boxed{-t - t^2}$$

$$u_2(t) = \int_{s=0}^{s=t} 2 ds = \left[2s \right]_{s=0}^{s=t} = \boxed{2t}$$

⇒ particular solution to the non-homogeneous system is:

$$\boxed{\bar{y}_p(t) = (-t - t^2) \begin{pmatrix} \bar{e}^t \\ \bar{e}^t \end{pmatrix} + 2t \begin{pmatrix} (t+1)\bar{e}^t \\ (t+\frac{1}{2})\bar{e}^t \end{pmatrix}}$$

(4) We now combine \bar{y}_c and \bar{y}_p and find the general solution to the non-homogeneous ODE system:

$$\bar{y}(t) = \bar{y}_c(t) + \bar{y}_p(t)$$

$$\bar{y}(t) = c_1 \left(\frac{e^{-t}}{e^t} \right) + c_2 \left(\frac{(t+1)e^{-t}}{(t+\frac{1}{2})e^t} \right) - (t+t^2) \left(\frac{e^{-t}}{e^t} \right) + 2t \left(\frac{(t+1)e^{-t}}{(t+\frac{1}{2})e^t} \right) \quad (*)$$

The final step is to use the IC for the non-homogeneous system to find c_1 and c_2 and hence the unique solution:

$$\text{IC: } \bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \stackrel{(*)}{\Rightarrow} \begin{pmatrix} c_1 + c_2 \\ c_1 + \frac{1}{2}c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 + \frac{1}{2}c_2 = 1 \end{cases}$$

Subtract: $\frac{1}{2}c_2 = 0 \Rightarrow c_2 = 0$

$$\Rightarrow c_1 + 0 = 1 \Rightarrow c_1 = 1$$

Therefore the unique solution to the non-homogeneous system is:

$$\bar{y}(t) = \begin{bmatrix} e^{-t} - (t+t^2)e^{-t} + 2t(t+1)e^{-t} \\ e^{-t} - (t+t^2)e^{-t} + 2t(t+\frac{1}{2})e^{-t} \end{bmatrix} \Rightarrow \boxed{\bar{y}(t) = \begin{bmatrix} (1+t+t^2)e^{-t} \\ (1+t^2)e^{-t} \end{bmatrix}}$$

(5) The unique solution is shown in the following figure.

