

Partial solutions to part (b) of HW # 4

Find the unique solution to $\bar{y}' = P \bar{y}$ with IC: $\bar{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$Q1. \quad P = \begin{pmatrix} 1 & 5/4 \\ 1 & 3 \end{pmatrix}$$

$$\text{ansatz: } \bar{y}(t) = \bar{r} e^{rt} \xrightarrow{\text{ODE}} P \bar{r} = r \bar{r}$$

$$\det(P - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & 5/4 \\ 1 & 3-r \end{vmatrix} = (1-r)(3-r) - \frac{5}{4} = 0 \Rightarrow r^2 - 4r + \frac{7}{4} = 0$$

$$\Rightarrow r_{1,2} = \frac{4 \pm \sqrt{16-7}}{2} = \frac{4 \pm 3}{2} \Rightarrow \boxed{r_1 = \frac{7}{2}, \quad r_2 = \frac{1}{2}} \Rightarrow \boxed{\text{origin is an unstable node.}}$$

$$r_1 = \frac{7}{2} \Rightarrow (P - r_1 I) \bar{r} = \overset{(1)}{0} \Rightarrow \begin{pmatrix} 1 - \frac{7}{2} & 5/4 \\ 1 & 3 - \frac{7}{2} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\frac{5}{2}Y_1 + \frac{5}{4}Y_2 = 0 \\ Y_1 - \frac{1}{2}Y_2 = 0 \end{cases}$$

both eqns are the same

$$\Rightarrow Y_1 - \frac{1}{2}Y_2 = 0 \Rightarrow \begin{cases} Y_1 = 1 \\ Y_2 = 2 \end{cases} \Rightarrow \boxed{\bar{Y}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$r_2 = \frac{1}{2} \Rightarrow \dots \Rightarrow \bar{Y}^{(2)} = \boxed{\begin{pmatrix} 1 \\ -2/5 \end{pmatrix}}$$

$$\Rightarrow \text{Two solution families: } \begin{cases} \bar{Y}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{\frac{7}{2}t} \\ \bar{Y}^{(2)} = \begin{pmatrix} 1 \\ -2/5 \end{pmatrix} e^{\frac{1}{2}t} \end{cases}$$

$$\text{Wronskian: } W[\bar{Y}^{(1)}, \bar{Y}^{(2)}](t) = \begin{vmatrix} e^{\frac{7}{2}t} & e^{\frac{1}{2}t} \\ 2e^{\frac{7}{2}t} & -\frac{2}{5}e^{\frac{1}{2}t} \end{vmatrix} = \left(-\frac{2}{5} - 2\right) e^{4t} = -\frac{12}{5} e^{4t} \neq 0$$

Therefore $\bar{Y}^{(1)}$ and $\bar{Y}^{(2)}$ form a set of fundamental solutions.

general solution: $\bar{y}(t) = c_1 \bar{Y}^{(1)}(t) + c_2 \bar{Y}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{\frac{7}{2}t} + c_2 \begin{pmatrix} 1 \\ -2/5 \end{pmatrix} e^{\frac{1}{2}t}$

$$\text{IC} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2/5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 - \frac{2}{5}c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ 2c_1 - \frac{2}{5}c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{7}{12} \\ c_2 = \frac{5}{12} \end{cases}$$

$$Q2. \quad P = \begin{pmatrix} 1 & 3/4 \\ -7 & -4 \end{pmatrix}$$

$$\text{ansatz: } \bar{y}(t) = \bar{Y} e^{rt} \xrightarrow{\text{ODE}} P\bar{Y} = r\bar{Y}$$

$$\det(P - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & 3/4 \\ -7 & -4-r \end{vmatrix} = 0 \Rightarrow (1-r)(-4-r) + \frac{21}{4} = 0 \Rightarrow r^2 + 3r + \frac{5}{4} = 0$$

$$\Rightarrow r_{1,2} = \frac{-3 \pm \sqrt{9-5}}{2} = \frac{-3 \pm 2}{2} \Rightarrow \boxed{r_1 = -\frac{1}{2}, \quad r_2 = -\frac{5}{2}} \Rightarrow \boxed{\text{origin is an asymptotically stable node.}}$$

$$r_1 = -\frac{1}{2} \Rightarrow (P - r_1 I) \bar{Y}^{(1)} = \bar{0} \Rightarrow \dots \Rightarrow \boxed{\bar{Y}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

$$r_2 = -\frac{5}{2} \Rightarrow (P - r_2 I) \bar{Y}^{(2)} = \bar{0} \Rightarrow \dots \Rightarrow \boxed{\bar{Y}^{(2)} = \begin{pmatrix} 1 \\ -\frac{14}{3} \end{pmatrix}}$$

$$\Rightarrow \text{Two solution families: } \begin{cases} \bar{Y}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-\frac{1}{2}t} \\ \bar{Y}^{(2)} = \begin{pmatrix} 1 \\ -\frac{14}{3} \end{pmatrix} e^{-\frac{5}{2}t} \end{cases}$$

$$\text{Wronskian: } W[\bar{Y}^{(1)}, \bar{Y}^{(2)}](t) = \begin{vmatrix} e^{-\frac{1}{2}t} & e^{-\frac{5}{2}t} \\ -2e^{-\frac{1}{2}t} & -\frac{14}{3}e^{-\frac{5}{2}t} \end{vmatrix} = \left(-\frac{14}{3}+2\right) e^{-3t} = -\frac{8}{3} e^{-3t} \neq 0$$

therefore $\bar{Y}^{(1)}$ and $\bar{Y}^{(2)}$ form a set of fundamental solutions.

$$\text{general solution: } \bar{y}(t) = c_1 \bar{Y}^{(1)}(t) + c_2 \bar{Y}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-\frac{1}{2}t} + c_2 \begin{pmatrix} 1 \\ -\frac{14}{3} \end{pmatrix} e^{-\frac{5}{2}t}$$

$$\text{IC} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -\frac{14}{3} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -2c_1 - \frac{14}{3}c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ -2c_1 - \frac{14}{3}c_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{17}{8} \\ c_2 = -\frac{9}{8} \end{cases} \Rightarrow \boxed{\bar{y}(t) = \begin{pmatrix} 17/8 \\ -17/4 \end{pmatrix} e^{-\frac{1}{2}t} + \begin{pmatrix} -9/8 \\ 21/4 \end{pmatrix} e^{-\frac{5}{2}t}}$$

Q3. $P = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

ansatz: $\bar{y}(t) = \bar{r} e^{rt} \xrightarrow{\text{ODE}} P\bar{y} = r\bar{y}$

$$\det(P - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & -1 \\ 2 & 3-r \end{vmatrix} = (1-r)(3-r) + 2 = 0 \Rightarrow r^2 - 4r + 5 = 0$$

$$\Rightarrow r_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm 2i}{2} \Rightarrow \boxed{r_1 = 2+i, r_2 = 2-i}$$

origin is an unstable spiral point

$$r_1 = 2+i \Rightarrow (P - r_1 I) \bar{y}^{(1)} = \bar{0} \Rightarrow \begin{pmatrix} 1-2-i & -1 \\ 2 & 3-2-i \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (r_1 - i)\bar{y}_1 - \bar{y}_2 = 0 \\ 2\bar{y}_1 + (1-i)\bar{y}_2 = 0 \end{cases}$$

both eqn's are the same.

$$\Rightarrow -(1+i)\bar{y}_1 - \bar{y}_2 = 0 \Rightarrow \begin{cases} \bar{y}_1 = 1 \\ \bar{y}_2 = -1-i \end{cases} \Rightarrow \boxed{\bar{y}^{(1)} = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}}$$

Since $r_2 = 2-i$ is the conjugate of $r_1 = 2+i$, $\bar{y}^{(2)}$ will also be the conjugate of $\bar{y}^{(1)}$ $\Rightarrow \boxed{\bar{y}^{(2)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}}$

\Rightarrow Two solution families: $\bar{y}^{(1)} = \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(2+i)t}$ and $\bar{y}^{(2)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(2-i)t}$

We know that $\bar{y}^{(1)} = \bar{u} + i\bar{v}$ and $\bar{y}^{(2)} = \bar{u} - i\bar{v}$. We first find \bar{u} and \bar{v} . To find \bar{u} and \bar{v} , it is enough to write $\bar{y}^{(1)}$ as:

$$\bar{y}^{(1)} = \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{2t} \cdot e^{it} = \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{2t} (\cos t + i \sin t) = \begin{pmatrix} e^{2t} \cos t + i e^{2t} \sin t \\ -e^{2t}(\cos t + i \sin t) \end{pmatrix}$$

$$\Rightarrow \bar{y}^{(1)} = \begin{pmatrix} e^{2t} \cos t + i e^{2t} \sin t \\ -e^{2t}(\cos t + i \sin t + i \cos t - \sin t) \end{pmatrix} = \begin{pmatrix} e^{2t} \cos t \\ -e^{2t}(\cos t - \sin t) \end{pmatrix} + i \begin{pmatrix} e^{2t} \sin t \\ -e^{2t}(\sin t + \cos t) \end{pmatrix}$$

$$\Rightarrow \bar{u} = e^{2t} \begin{pmatrix} \cos t \\ \sin t - \cos t \end{pmatrix} \quad \text{and} \quad \bar{v} = e^{2t} \begin{pmatrix} \sin t \\ -\sin t - \cos t \end{pmatrix}$$

Wronskian: $W[\bar{u}, \bar{v}](t) = \begin{vmatrix} e^{2t} \cos t & e^{2t} \sin t \\ e^{2t}(\sin t - \cos t) & -e^{2t}(\sin t + \cos t) \end{vmatrix}$

Q3, cont.

$$\Rightarrow W[\bar{u}, \bar{v}](t) = e^{4t} \left(-\cancel{\cos t \sin t} - \cos^2 t - \sin^2 t + \cancel{\cos t \sin t} \right) = -e^{4t} \neq 0$$

$\Rightarrow \bar{u}$ and \bar{v} form a set of fundamental solutions.

general solution: $y(t) = c_1 \bar{u} + c_2 \bar{v} = c_1 e^{2t} \begin{pmatrix} \cos t \\ \sin t - \cos t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin t \\ -\sin t - \cos t \end{pmatrix}$

$$\text{IC} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 - c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 1 \\ c_1 + c_2 = -1 \end{cases}$$

\downarrow

$$c_2 = -2$$

\Rightarrow unique solution:

$$\boxed{\bar{y}(t) = e^{2t} \begin{pmatrix} \cos t \\ \sin t - \cos t \end{pmatrix} - 2e^{2t} \begin{pmatrix} \sin t \\ -\sin t - \cos t \end{pmatrix}}$$

$$Q4. \quad P = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

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$$\text{ansatz: } \bar{y}(t) = \bar{Y} e^{rt} \xrightarrow{\text{ODE}} P\bar{Y} = r\bar{Y}$$

$$\det(P - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & -2 \\ 1 & -1-r \end{vmatrix} = -(1-r)(1+r) + 2 = 0 \Rightarrow -1 + r^2 + 2 = 0 \Rightarrow r^2 = -1$$

$$\Rightarrow \boxed{r_1 = i, \quad r_2 = -i} \Rightarrow \boxed{\text{origin is a stable center} \\ (\text{it is asymptotically unstable})}$$

$$r_1 = i \Rightarrow (P - r_1 I) \bar{Y}^{(1)} = \bar{0} \Rightarrow \begin{pmatrix} 1-i & -2 \\ 1 & -1-i \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (1-i)Y_1 - 2Y_2 = 0 \\ Y_1 + (1+i)Y_2 = 0 \end{cases}$$

both eqn's are the same

$$\Rightarrow (1-i)Y_1 - 2Y_2 = 0 \Rightarrow \boxed{\bar{Y}^{(1)} = \begin{pmatrix} 1 \\ 1-i \\ 2 \end{pmatrix}} \Rightarrow \boxed{\bar{Y}^{(2)} = \begin{pmatrix} 1 \\ 1+i \\ 2 \end{pmatrix}}$$

$$\Rightarrow \text{Two solution families are } \bar{y}^{(1)} = \begin{pmatrix} 1 \\ 1-i \\ 2 \end{pmatrix} e^{it} \quad \text{and} \quad \bar{y}^{(2)} = \begin{pmatrix} 1 \\ 1+i \\ 2 \end{pmatrix} e^{-it}$$

$$\text{We need to find } \bar{u} \text{ and } \bar{v} \text{ where } \bar{y}^{(1)} = \bar{u} + i\bar{v} \text{ and } \bar{y}^{(2)} = \bar{u} - i\bar{v}.$$

we write $\bar{y}^{(1)}$ as:

$$\bar{y}^{(1)} = \begin{pmatrix} 1 \\ 1-i \\ 2 \end{pmatrix} (cost + i \sin t) = \begin{pmatrix} cost + i \sin t \\ \frac{1}{2}(1-i)(cost + i \sin t) \\ \frac{1}{2}cost + \frac{1}{2}i \sin t - \frac{1}{2}i(cost + \frac{1}{2} \sin t) \end{pmatrix} = \begin{pmatrix} cost + i \sin t \\ \frac{1}{2}cost + \frac{1}{2}i \sin t - \frac{1}{2}i(cost + \frac{1}{2} \sin t) \\ \frac{1}{2}cost + \frac{1}{2}i \sin t - \frac{1}{2}cost - \frac{1}{2}\sin t \end{pmatrix}$$

$$\Rightarrow \bar{y}^{(1)} = \begin{pmatrix} cost \\ \frac{1}{2}(cost + \sin t) \\ \frac{1}{2}cost + \frac{1}{2}\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \frac{1}{2}(\sin t - cost) \\ \frac{1}{2}\sin t - cost \end{pmatrix} \Rightarrow \bar{u} = \begin{pmatrix} cost \\ \frac{1}{2}(cost + \sin t) \\ \frac{1}{2}cost + \frac{1}{2}\sin t - cost \end{pmatrix} \text{ and } \bar{v} = \begin{pmatrix} \sin t \\ \frac{1}{2}(\sin t - cost) \\ \frac{1}{2}\sin t - cost \end{pmatrix}$$

$$\text{Wronskian: } W[\bar{u}, \bar{v}](t) = \begin{vmatrix} cost & \sin t \\ \frac{1}{2}(cost + \sin t) & \frac{1}{2}(\sin t - cost) \\ \frac{1}{2}cost + \frac{1}{2}\sin t - cost & \frac{1}{2}\sin t - cost \end{vmatrix} = \frac{1}{2}cost\sin t - \frac{1}{2}cost^2 - \frac{1}{2}cost\sin t - \frac{1}{2}\sin^2 t$$

$\Rightarrow W = -\frac{1}{2} \neq 0 \Rightarrow \bar{u} \text{ and } \bar{v} \text{ form a set of fundamental solutions.}$

$$\text{general solution: } \bar{y}(t) = c_1 \bar{u}(t) + c_2 \bar{v}(t) = c_1 \begin{pmatrix} cost \\ \frac{1}{2}(cost + \sin t) \\ \frac{1}{2}cost + \frac{1}{2}\sin t - cost \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \frac{1}{2}(\sin t - cost) \\ \frac{1}{2}\sin t - cost \end{pmatrix}$$

$$\text{IC} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} c_1 \\ \frac{1}{2}c_1 - \frac{1}{2}c_2 \\ \frac{1}{2}c_1 + \frac{1}{2}c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 1 \\ \frac{1}{2}c_1 - \frac{1}{2}c_2 = 1 \\ \frac{1}{2}c_1 + \frac{1}{2}c_2 = 1 \end{cases} \Rightarrow c_2 = -1$$

$$\Rightarrow \text{unique solution: } \boxed{\bar{y}(t) = \begin{pmatrix} cost + \sin t \\ cost \\ cost \end{pmatrix}}$$