

HW # 4 (partial solutions)

$$\textcircled{1} \quad \left\{ \begin{array}{l} \text{ODE: } y'' + y' = 0 \quad t \in [0, 10] \\ \text{IC1: } y(0) = 2 \\ \text{IC2: } y'(0) = 2 \end{array} \right.$$

a) 1. ansatz  $y(t) = e^{rt}$

2. ODE  $\Rightarrow (r^2 + r) e^{rt} = 0 \quad \xrightarrow{e^{rt} \neq 0} r^2 + r = 0$

3.  $r(r+1) = 0 \Rightarrow \begin{cases} r_1 = 0 \\ r_2 = -1 \end{cases}$

4. solution families  $\begin{cases} y_1 = e^{r_1 t} = e^0 = 1 \\ y_2 = e^{r_2 t} = e^{-t} \end{cases}$

5. general solution  $y(t) = c_1 \cdot 1 + c_2 e^{-t}$

6. IC1:  $y(0) = 2 \Rightarrow c_1 \cdot 1 + c_2 \cdot e^0 = 2 \Rightarrow c_1 + c_2 = 2$

IC2:  $y'(0) = 2$

$$y'(t) = -c_2 e^{-t} \Rightarrow -c_2 \cdot e^0 = 2 \Rightarrow c_2 = -2$$

$$\Rightarrow c_1 = 2 - c_2 = 2 + 2 = 4$$

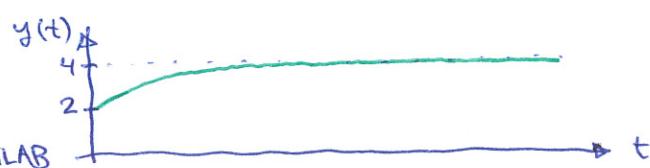
7. unique solution: 
$$y(t) = 4 - 2e^{-t}$$

b)  $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} 4 - 2e^{-t} = 4 - 2 \cdot \lim_{t \rightarrow \infty} e^{-t} = 4 - 2 \cdot 0 = 4$

c) you need to use MATLAB to plot the solution.

It will look like

this needs to be done in MATLAB



(2)

$$\textcircled{2} \text{ ODE: } t^2 y'' - t(t+2) y' + (t+2) y = 0, \quad t > 0$$

$$\begin{cases} y_1 = t \\ y_2 = t \cdot e^t \end{cases}$$


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1) need to show that  $y_1$  satisfies the ODE:

$$\left. \begin{array}{l} y_1 = t \\ y'_1 = 1 \\ y''_1 = 0 \end{array} \right\} \xrightarrow{\text{ODE}} t^2 \cdot 0 - t(t+2) \cdot 1 + (t+2)t = 0 - t^2 - 2t + t^2 + 2t = 0$$

2) need to show that  $y_2$  satisfies the ODE:

$$\left. \begin{array}{l} y_2 = t e^t \\ y'_2 = e^t + t e^t = (1+t) e^t \\ y''_2 = e^t + (1+t) e^t = (2+t) e^t \end{array} \right\} \xrightarrow{\text{ODE}} \left[ \underbrace{t^2(2+t) - t(t+2)(1+t) + (t+2)t}_{(2+t)(t^2 - t - t^2 + t)} \right]_{e=0} = 0$$

3) need to show that the Wronskian of  $y_1$  and  $y_2$  is non-zero for  $t > 0$ :

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} t & t \cdot e^t \\ 1 & (1+t) e^t \end{vmatrix} = (t(1+t) - t) e^t = t^2 e^t \neq 0$$

because  $t > 0$ .

$$\begin{cases} \text{ODE: } y'' + 4y' + 5y = 0 & t \in [0, 5] \\ \text{IC1: } y(0) = 1 \\ \text{IC2: } y'(0) = 2 \end{cases}$$

a) 1. ansatz  $y(t) = e^{rt}$

2. ODE  $\Rightarrow e^{rt} (r^2 + 4r + 5) = 0 \xrightarrow{e^{rt} \neq 0} r^2 + 4r + 5 = 0$

3.  $r_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i \Rightarrow \begin{cases} r_1 = -2 + i \\ r_2 = -2 - i \end{cases}$

4. solution families  $\begin{cases} y_1 = e^{(-2+i)t} = e^{-2t} (\cos t + i \sin t) \\ y_2 = e^{(-2-i)t} = e^{-2t} (\cos t - i \sin t) \end{cases}$

It is OK if you skip this step.

5. two fundamental solutions:  $\begin{cases} y_1 = e^{-2t} \cdot \cos t \\ y_2 = e^{-2t} \cdot \sin t \end{cases}$

these two solutions are fundamental solutions because they satisfy the ODE and because their Wronskian is non-zero.

6. general solution:  $y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

7.  $\begin{cases} \text{IC1: } y(0) = 1 \Rightarrow c_1 \overset{0}{e^{-2t}} \cos 0 + c_2 \overset{0}{e^{-2t}} \sin 0 = \boxed{c_1 = 1} \\ \text{IC2: } y'(0) = 2 \end{cases}$

$$y'(t) = -2c_1 \overset{-2t}{e} \cos t - c_1 \overset{-2t}{e} \sin t - 2c_2 \overset{-2t}{e} \sin t + c_2 \overset{-2t}{e} \cos t$$

$$\Rightarrow -2c_1 \overset{0}{e} \cos 0 - c_1 \overset{0}{e} \sin 0 - 2c_2 \overset{0}{e} \sin 0 + c_2 \overset{0}{e} \cos 0 = 2$$

$$\Rightarrow -2c_1 + c_2 = 2 \Rightarrow -2 \times 1 + c_2 = 2 \Rightarrow \boxed{c_2 = 4}$$

8) unique solution:  $y(t) = e^{-2t} (\cos t + 4 \sin t)$

b) As  $t \rightarrow \infty$ , the term  $e^{-2t} \rightarrow 0$ . Since  $\cos t + 4 \sin t$  does not grow then  $\lim_{t \rightarrow \infty} y(t) = 0$ .

c) you need to use MATLAB!

④  $\left\{ \begin{array}{l} \text{ODE: } y'' + 2y' + y = \cos t \quad t \in [0, 60] \\ \text{IC1: } y(0) = 1 \\ \text{IC2: } y'(0) = 0 \end{array} \right.$

a)

the complementary solution:

1) ansatz:  $y = e^{rt}$

2) ODE  $\Rightarrow e^{rt}(r^2 + 2r + 1) = 0 \Rightarrow r^2 + 2r + 1 = 0$

3)  $r_{1,2} = \frac{-2 \pm \sqrt{4 - 4}}{2} = -1 \Rightarrow r_1 = r_2 = -1$

4) fundamental solutions are  $\left\{ \begin{array}{l} y_1 = e^{-t} \\ y_2 = t \cdot e^{-t} \end{array} \right.$

5) complementary solution: 
$$\boxed{y_c(t) = c_1 e^{-t} + c_2 t e^{-t}}$$

a particular solution:

we use the method of variation of parameters:

1)  $y_p(t) = u_1(t) \cdot y_1(t) + u_2(t) \cdot y_2(t)$  where  $\left\{ \begin{array}{l} y_1(t) = e^{-t} \\ y_2(t) = t \cdot e^{-t} \end{array} \right.$

$$y'_p(t) = u_1 \cdot y'_1 + u_2 \cdot y'_2 + \boxed{u'_1 y_1 + u'_2 y_2}$$

we let this term be zero

$$\boxed{u'_1 y_1 + u'_2 y_2 = 0} \quad ①$$

$$y''_p(t) = u'_1 \cdot y'_1 + u'_2 \cdot y'_2 + u_1 \cdot y''_1 + u_2 \cdot y''_2$$

2) we plug  $y_p, y'_p, y''_p$  into the ODE:

$$[u'_1 \cdot y'_1 + u'_2 \cdot y'_2 + \underline{u_1 \cdot y''_1} + \underline{u_2 \cdot y''_2}] + 2[u_1 \cdot y'_1 + u_2 \cdot y'_2] + [u_1 \cdot y_1 + u_2 \cdot y_2] = \cos t$$

$$\Rightarrow u_1(\underbrace{y''_1 + 2y'_1 + y_1}_0) + u_2(\underbrace{y''_2 + 2y'_2 + y_2}_0) + u'_1 y'_1 + u'_2 y'_2 = \cos t$$

$$\Rightarrow \boxed{u'_1 y'_1 + u'_2 y'_2 = \cos t} \quad (2)$$

3) Now we solve ① and ② by solving the system:

$$\begin{cases} u'_1 y_1 + u'_2 y_2 = 0 \\ u'_1 y'_1 + u'_2 y'_2 = \cos t \end{cases}$$

after some work  
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$$\begin{cases} u'_1(t) = \frac{-y_2(t) \cdot \cos t}{y'_2 y_1 - y_2 y'_1} \\ u'_2(t) = \frac{y_1(t) \cdot \cos t}{y'_2 y_1 - y_2 y'_1} \end{cases}$$

$$\begin{cases} y_1 = e^{-t} \Rightarrow y'_1 = -e^{-t} \\ y_2 = t e^{-t} \Rightarrow y'_2 = (1-t)e^{-t} \end{cases}$$

$$\text{Here, } W[y_1, y_2] = y'_2 y_1 - y_2 y'_1 = (1-t)e^{-t} \cdot e^{-t} - t \cdot e^{-t}(-e^{-t}) = \\ = (1-t+t)e^{-2t} = e^{-2t}$$

$$\Rightarrow \begin{cases} u'_1(t) = \frac{-t \cdot e^{-t} \cdot \cos t}{e^{-2t}} = -t \cdot e^t \cdot \cos t \\ u'_2(t) = \frac{e^{-t} \cdot \cos t}{e^{-2t}} = e^t \cdot \cos t \end{cases}$$

$$4) \begin{cases} u_1(t) = \int_{s=0}^{s=t} -s \cdot e^s \cdot \cos s \cdot ds = \left\{ \text{using integral table} \right\} = \frac{-e^t}{2} (t \cos t + (t-1) \sin t) \\ u_2(t) = \int_{s=0}^{s=t} e^s \cdot \cos s \cdot ds = \left\{ \text{using integral table} \right\} = \frac{e^t}{2} (\cos t + \sin t) - \frac{1}{2} \end{cases}$$

$$5) \text{ particular solution: } y_p(t) = u_1 e^{-t} + u_2 t e^{-t}$$

$$\Rightarrow y_p(t) = \frac{1}{2} (t \cos t + (t-1) \sin t) + \frac{t}{2} (\cos t + \sin t) - \frac{1}{2} t e^{-t}$$

$$\Rightarrow \boxed{y_p(t) = \frac{1}{2} \sin t - \frac{t}{2} e^{-t}}$$

general solution

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1 \cdot e^{-t} + c_2 \cdot t \cdot e^{-t} + \frac{1}{2} \sin t - \frac{t}{2} e^{-t}$$

To find  $c_1$  and  $c_2$ :

$$\underline{\underline{IC1}}: \quad y(0) = 1 \Rightarrow c_1 \cdot e^0 + c_2 \cdot 0 + \frac{1}{2} \sin 0 - 0 = 1 \\ \Rightarrow \boxed{c_1 = 1}$$

$$\underline{\underline{IC2}}: \quad y'(0) = 0$$

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + \frac{1}{2} \cos t - \frac{1}{2} e^{-t} + \frac{t}{2} e^{-t} \\ \Rightarrow -c_1 e^0 + c_2 e^0 - 0 + \frac{1}{2} \cos 0 - \frac{1}{2} e^0 + 0 = 0 \\ \Rightarrow -c_1 + c_2 + \frac{1}{2} - \frac{1}{2} = 0 \Rightarrow c_2 = c_1 \Rightarrow \boxed{c_2 = 1}$$

unique solution:

$$\boxed{y(t) = e^{-t} + t \cdot e^{-t} + \frac{1}{2} \sin t - \frac{t}{2} e^{-t}}$$

b) you need to use Matlab to plot the graph of the solution.