

$$\textcircled{1} \quad \begin{cases} \text{ODE:} & y' = -y + \cos t + 4 \\ \text{IC:} & y(0) = 0 \end{cases} \quad t \geq 0$$

a) Let $L(y) = y' + y$. We need to show $L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$

$$\begin{aligned} L(c_1 y_1 + c_2 y_2) &= (c_1 y_1 + c_2 y_2)' + (c_1 y_1 + c_2 y_2) \\ &= c_1 y_1' + c_2 y_2' + c_1 y_1 + c_2 y_2 \end{aligned}$$

$$\begin{aligned} c_1 L(y_1) + c_2 L(y_2) &= c_1 (y_1' + y_1) + c_2 (y_2' + y_2) \\ &= c_1 y_1' + c_1 y_1 + c_2 y_2' + c_2 y_2 \end{aligned}$$

Hence $L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$ and the ODE is linear.

b) Since the ODE is linear, we write it in the form

$$y' + p(t)y = g(t) \quad \text{where} \quad \begin{cases} p(t) = 1 \\ g(t) = \cos t + 4 \end{cases}$$

Since both $p(t)$ and $g(t)$ are continuous functions for every $t \in \mathbb{R}$, the IVP has a unique solution.

c) The method of integrating factor:

$$\mu(t) = e^{\int_0^t p(s) ds} = e^{\int_0^t ds} = e^t$$

$$y(t) = \frac{\int_0^t \mu(s) g(s) ds + C}{\mu(t)}$$

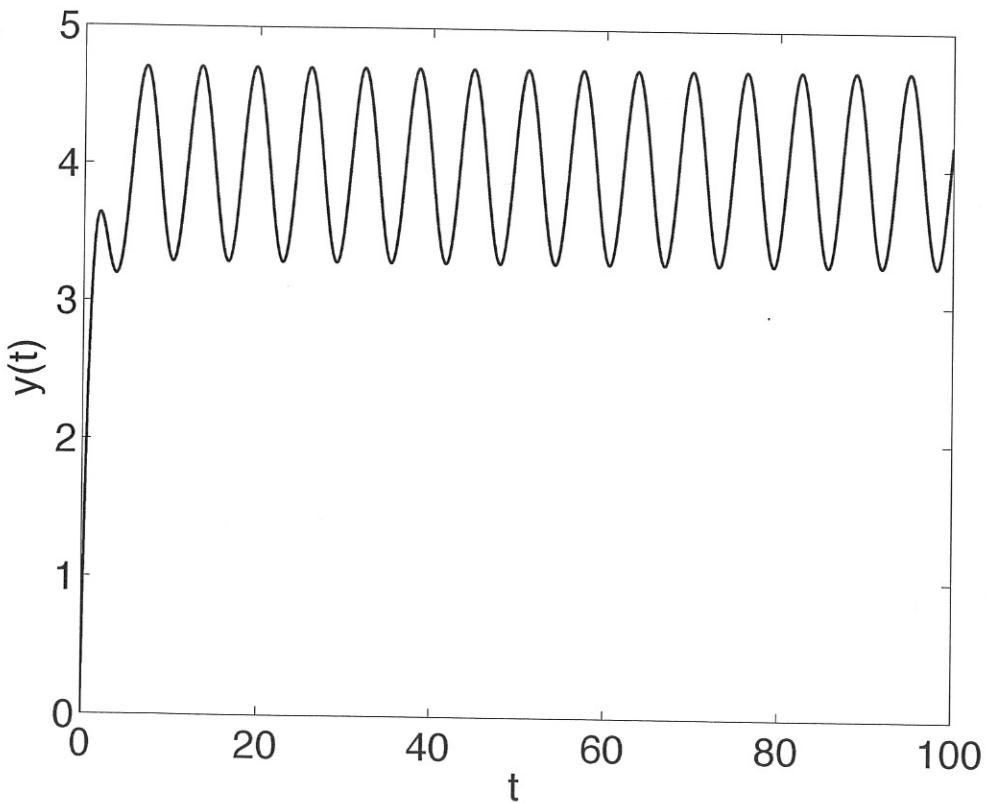
$$\text{where by IC: } y(0) = 0 \Rightarrow 0 = \frac{0 + C}{\mu(0)} = \frac{C}{e^0} = C \Rightarrow C = 0$$

$$\Rightarrow y(t) = \frac{\int_0^t \mu(s) g(s) ds}{\mu(t)} = \bar{e}^{-t} \int_0^t e^s (\cos s + 4) ds$$

$$= \bar{e}^{-t} \left(4 \int_0^t e^s ds + \int_0^t e^s \cos s ds \right) =$$

$$\begin{aligned}
 \Rightarrow y(t) &= 4e^{-t} \left[e^s \right]_{s=0}^{s=t} + e^{-t} \left[\frac{e^s}{1+1} (\cos s + \sin s) \right]_{s=0}^{s=t} \\
 &= 4e^{-t} [e^t - 1] + \frac{1}{2} e^{-t} [e^t (\cos t + \sin t) - 1] \\
 &= 4 - 4e^{-t} + \frac{1}{2} \cos t + \frac{1}{2} \sin t - \frac{1}{2} e^{-t} \\
 \Rightarrow y(t) &= 4 - \frac{9}{2} e^{-t} + \frac{1}{2} \cos t + \frac{1}{2} \sin t \quad (*) \tag{2}
 \end{aligned}$$

- d) By (*), as $t \rightarrow \infty$, $y(t)$ will oscillate about 4, because $\lim_{t \rightarrow \infty} e^{-t} = 0$ and $\frac{1}{2}(\cos t + \sin t)$ is oscillatory about zero.
- e) Here is the plot of the solution (*) for $t \in [0, 100]$:



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(2) $\left\{ \begin{array}{l} \text{ODE: } y' = \frac{2x}{y+x^2y} \\ \text{IC: } y(0) = -2 \end{array} \right.$

a) The ODE is nonlinear since if we write it in the form

$y' = f(x, y) = \frac{2x}{y(1+x^2)}$, then $f(x, y)$ is nonlinear in y because of the $\frac{1}{y}$ term.

b) we need to study the continuity of the following two functions:

$$f(x, y) = \frac{2x}{y(1+x^2)} \quad \text{and} \quad \partial_y f(x, y) = \frac{2x}{1+x^2} \cdot \frac{-1}{y^2}$$

Both functions are continuous in x everywhere.

Both functions are continuous in y everywhere except at $y=0$.

Therefore by the theorem on the existence & uniqueness of the solution of nonlinear IVP's (Theorem 2.4.2), there exists a unique solution around the initial point $x=0, y=-2$.

c) The ODE is separable, because it can be written in the form

$$N(y) \cdot y' = M(x) \quad \text{where} \quad \left\{ \begin{array}{l} N(y) = y \\ M(x) = \frac{2x}{1+x^2} \end{array} \right.$$

d) Since the ODE is separable, we can integrate both sides of the equation $N(y) \cdot y' = M(x)$ as follows:

$$\int_{-2}^y N(s) ds = \int_0^x M(s) ds \Rightarrow \int_{-2}^y s ds = \int_0^x \frac{2s}{1+s^2} ds$$

$$\Rightarrow \left[\frac{s^2}{2} \right]_{s=-2}^{s=y} = \left[\ln(1+s^2) \right]_{s=0}^{s=x}$$

$$\Rightarrow \frac{y^2}{2} - 2 = \ln(1+x^2) \Rightarrow y^2 = 4 + 2 \cdot \ln(1+x^2)$$

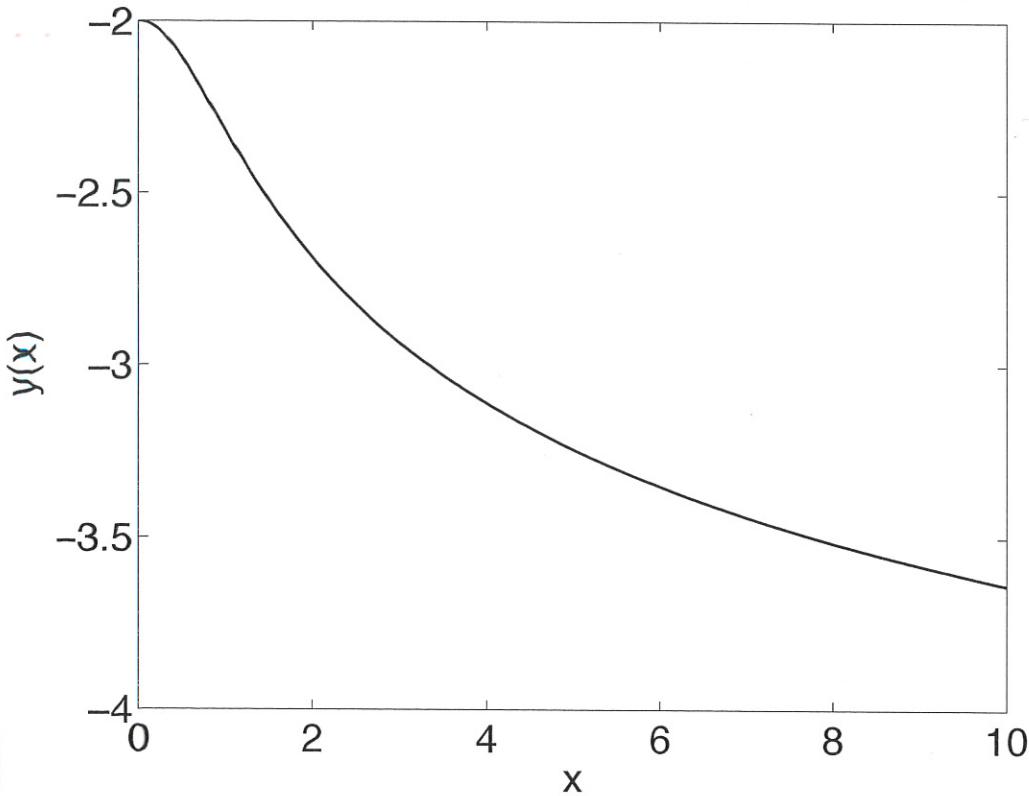
$$\Rightarrow y = \pm \sqrt{4 + 2 \cdot \ln(1+x^2)}$$

Since $1+x^2 \geq 1 \quad \forall x \in \mathbb{R}$, then $\ln(1+x^2) \geq 0$ is well defined.

Hence, under the radical is always positive and well defined.

Moreover, out of the two solutions with \pm signs, only the negative sign satisfies the IC $\Rightarrow \boxed{y = -\sqrt{4 + 2 \cdot \ln(1+x^2)}} \quad (\Delta)$

e) Here is the plot of the solution (Δ) for $x \in [0, 10]$:



③ $\left\{ \begin{array}{l} \text{ODE: } y' = \frac{1 - \frac{y^2}{x}}{2y \ln x} \\ \text{IC: } y(2) = 1 \end{array} \right.$

a) The ODE is nonlinear since if we write it in the form

$y' = f(x, y) = \frac{1 - \frac{y^2}{x}}{2y \ln x}$, then $f(x, y)$ is nonlinear in y because of the terms y^2 and $\frac{1}{y}$.

b) we need to study the continuity of the following two functions:

$$f(x, y) = \frac{1 - \frac{y^2}{x}}{2y \ln x} \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = \frac{-1 - \frac{y^2}{x}}{2y^2 \ln x}$$

For both functions to be well defined and continuous, we need:

- 1) $x \neq 0$ (because of the term $\frac{1}{x}$)
- 2) $x > 0$ (because of the term $\ln x$)
- 3) $\ln x \neq 0 \Rightarrow x \neq 1$ (because of the term $\frac{1}{\ln x}$)
- 4) $y \neq 0$ (because of the term $\frac{1}{y}$)

Since at the initial point $x=2, y=1$ and its neighborhood, both functions are well defined and continuous, then by theorem 2.4.2 there exists a unique solution around $x=2, y=1$.

c) In order to show that the ODE is exact, we first write it in the form

$$N(x, y) \cdot y' = M(x, y) \quad \text{where} \quad \left\{ \begin{array}{l} M(x, y) = 1 - \frac{y^2}{x} \\ N(x, y) = 2y \ln x \end{array} \right.$$

We then check if $M_y + N_x = 0$:

$$\left. \begin{array}{l} M_y = -\frac{2y}{x} \\ N_x = \frac{2y}{x} \end{array} \right\} \Rightarrow M_y + N_x = 0 \Rightarrow \text{The ODE is exact.}$$

d) Since the ODE is exact, there exists a function $\Psi(x,y)$ such that:

$$\left\{ \begin{array}{l} \Psi_x = -M = \frac{y^2}{x} - 1 \\ \Psi_y = N = 2y \ln x \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \Psi_x = -M = \frac{y^2}{x} - 1 \\ \Psi_y = N = 2y \ln x \end{array} \right. \quad (2)$$

$$(1) \Rightarrow \Psi(x,y) = \int \left(\frac{y^2}{x} - 1 \right) dx \Rightarrow \left\{ \text{since } x > 0 \Rightarrow \ln|x| = \ln x \right\}$$

$$\Rightarrow \Psi(x,y) = y^2 \ln x - x + c(y)$$

$$(2) \Rightarrow 2y \cdot \ln x + c'(y) = 2y \ln x \Rightarrow c'(y) = 0 \Rightarrow c(y) = c_0$$

$$\Rightarrow \Psi(x,y) = y^2 \ln x - x + c_0$$

\Rightarrow The solution is implicitly given by the equation $\Psi(x,y) = c_1$

$$\text{which reads } y^2 \ln x - x + c_0 = c_1 \Rightarrow y^2 \ln x - x = \underbrace{c_1 - c_0}_{c_2}$$

$$\text{By IC: } y(2) = 1 \Rightarrow 1 \cdot \ln 2 - 2 = c_2$$

\Rightarrow Implicit form of the solution

$$y^2 \ln x - x = \ln 2 - 2 \quad (\star)$$

To find the explicit form of the solution, we solve (\star) for y :

$$y = \pm \sqrt{\frac{x + \ln 2 - 2}{\ln x}}. \quad \text{For } x \geq 2, \text{ under the radical is positive.}$$

$$\text{Moreover, only the positive sign satisfies the IC.} \Rightarrow y = \sqrt{\frac{x + \ln 2 - 2}{\ln x}}$$

e) Here are the plots of the solution obtained by both explicit & implicit forms.

