HW 1

(MATH 316)

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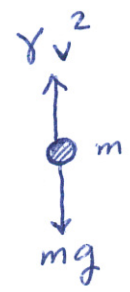
① Mathematical modeling of a falling object.

a) We will use Newton's law to derive an ODE for the velocity v of a falling object of mass m : $\boxed{F = m \frac{dv}{dt}}$



total force : $F = \underbrace{mg}_{\text{gravitational force}} - \underbrace{\gamma v^2}_{\text{drag force}}$

Here, γ is the drag coefficient.

Hence, we have : $\boxed{m \frac{dv(t)}{dt} = mg - \gamma . v(t)^2}$ (*)

b) To determine the equilibrium velocity, we let $\frac{dv(t)}{dt} = 0$.

From the ODE (*), we then obtain :

$$mg - \gamma . v^2 = 0 \Rightarrow v^2 = \frac{mg}{\gamma}$$

assuming the equilibrium velocity is positive in downward direction $\Rightarrow$ $\boxed{v_E = \sqrt{\frac{mg}{\gamma}}}$ (▨)

c) $g = 9.8 \left[\frac{m}{s^2}\right]$, $m = 10$ [Kg] , $v_E = 49 \left[\frac{m}{s}\right]$

In order to find the drag coefficient γ, we use (▨) :

$$v_E = \sqrt{\frac{mg}{\gamma}} \Rightarrow \gamma = \frac{mg}{v_E^2} = \frac{10 \times 9.8}{49^2} \simeq 0.04$$

(2) Classification of ODE's.

a) $y'' = 2y + \sin t$

The ODE is 2nd order because of the term y''.

The ODE is linear, because if we let $L(y) = y'' - 2y$,

$$L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)'' - 2(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' - 2c_1 y_1 - 2c_2 y_2$$

$$c_1 L(y_1) + c_2 L(y_2) = c_1 (y_1'' - 2y_1) + c_2 (y_2'' - 2y_2)$$

$$= c_1 y_1'' - 2c_1 y_1 + c_2 y_2'' - 2c_2 y_2$$

$\Rightarrow$ Since $L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$, the ODE is linear.

b) $y'' = \sin y + 2t$

The ODE is 2nd order because of the term y''.

The ODE is nonlinear because $\sin y$ is nonlinear in y.

c) $y y' + t^2 = 0$

The ODE is 1st order because of the term y'.

The ODE is nonlinear because yy' is nonlinear in y and y'.

③ Solving Simple ODE's.

$Q(t)$: the amount of a radioactive material at time t

r : the decay constant

$$Q'(t) = -r\,Q(t), \qquad r > 0$$

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a) $\begin{cases} t=0 \text{ [day]} \Rightarrow Q(t) = 100 \text{ [mg]} \\ t=7 \text{ [day]} \Rightarrow Q(t) = 80 \text{ [mg]} \end{cases} \Rightarrow r = ?$

We write the ODE in the form $\dfrac{Q'(t)}{Q(t)} = -r$

Since $Q \geq 0 \Rightarrow \dfrac{d}{dt}\left(\ln Q(t)\right) = -r$

integrate in $t \Rightarrow \ln Q(t) = -rt + C$

take exponential $\Rightarrow Q(t) = e^{C} \cdot e^{-rt}$

$\{e^{C} = C_0\} \Rightarrow Q(t) = C_0 \cdot e^{-rt}$ where C_0 is a constant.

where C_0 is a constant obtained by setting $t=0 \Rightarrow Q(0) = C_0 \cdot e^{-r\times 0} = C_0$

Hence: $\boxed{Q(t) = Q(0)\, e^{-rt}}$ (*)

b) We have: $\begin{cases} t=0 \text{ [days]} \Rightarrow Q(t) = 100 \text{ [mg]} \\ t=7 \text{ [days]} \Rightarrow Q(t) = 80 \text{ [mg]} \end{cases}$

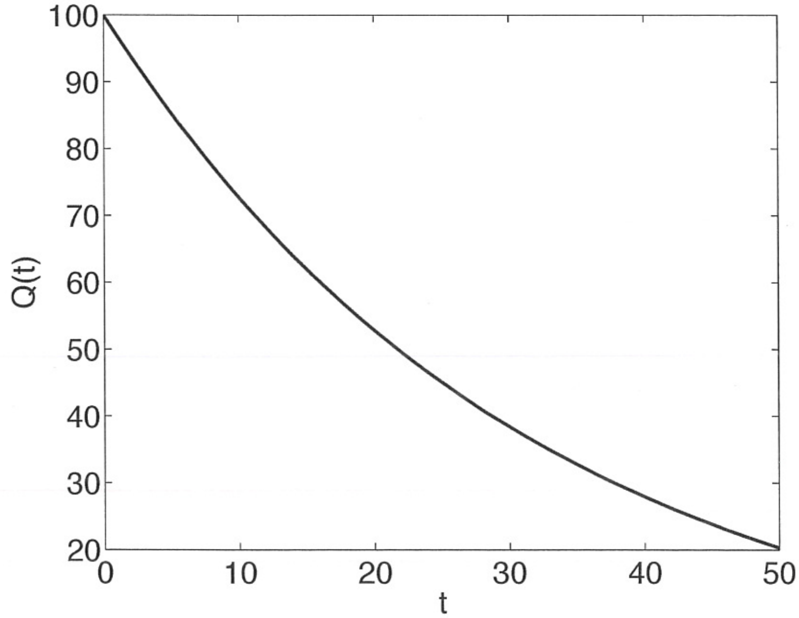
To find r, we use $\begin{cases} Q(0) = 100 \\ Q(7) = 80 \end{cases} \xrightarrow{(*)} 80 = 100 \cdot e^{-7r}$

$$\Longrightarrow r = \frac{-\ln 0.8}{7} \simeq 0.032$$

The unit of r: $[r] = \dfrac{[Q']}{[Q]} = \dfrac{\left[\frac{mg}{day}\right]}{[mg]} = \left[\dfrac{1}{day}\right]$

c) Here is the plot of the solution $Q(t) = 100 \cdot e^{-r \cdot t}$ with $r = \frac{-\ln 0.8}{7}$

for $t \in [0, 50]$:



d) $\left\{ \begin{array}{lll} t = 0 & \Rightarrow & Q(0) \\ t = ? & \Rightarrow & Q(t) = \frac{Q(0)}{2} \end{array} \right\} \xrightarrow{Q(t) = Q(0) \cdot e^{-rt}} \frac{Q(0)}{2} = Q(0) \cdot e^{-rt}$

$\Rightarrow \frac{1}{2} = e^{-rt} \Rightarrow t = \frac{-\ln \frac{1}{2}}{r} = \frac{-\ln 0.5}{\frac{-\ln 0.8}{7}} \simeq 21.7$ [day]

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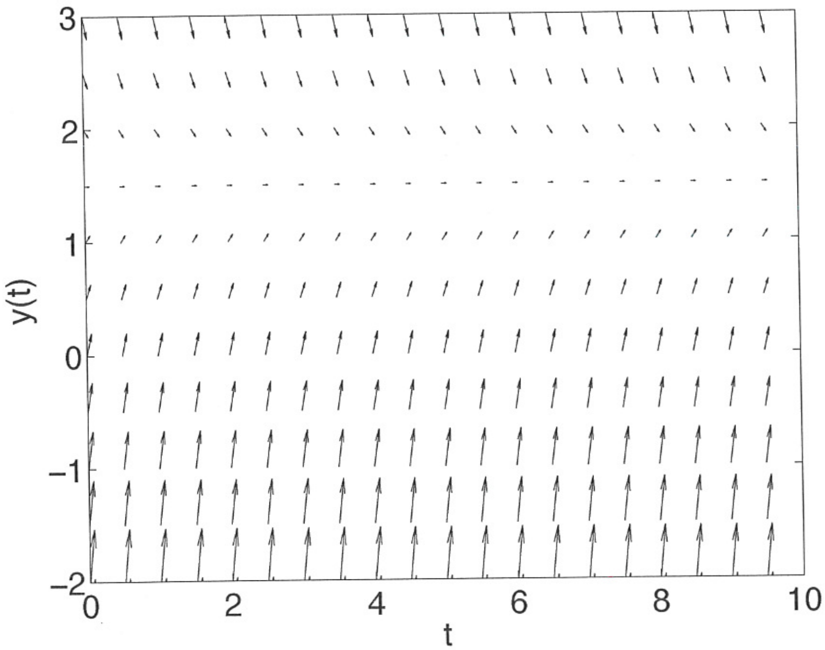
④ Direction fields.

a) To draw a direction field of the ODE $y'(t) = 3 - 2y(t)$, we need to choose a set of points (t, y) in ty-plane, and then at each point (t, y) we calculate y'.

The direction field is shown in the following figure.



b) $\lim_{t\to\infty} y(t)$ is the limiting solution, which is the equilibrium solution provided it exists. To find the equilibrium solution, we let $y' = 0 \Rightarrow 3 - 2y = 0 \Rightarrow y_E = \frac{3}{2}$.

We also observe from the direction field that all solutions converge to the limiting solution $y_E = \frac{3}{2}$ as t increases.

c) The behavior of the limiting solution does <u>not</u> depend on the initial solution $y(0)$, because no matter the starting point is, the solution always converge to the same limiting solution y_E.

$\boxed{\text{REMARK}}$

In order to find appropriate bounds on the y-axis of direction field, we notice that the limiting solution is $y_E = 1.5$. Therefore, a good choice is for instance $y \in [-4, 5]$ which contains $y_E = 1.5$.