Theorems of ch. 3

T1 If V is a vector space and $\mathbf{x}$ is an arbitrary element of V, then

(i) $0\mathbf{x} = \mathbf{0}$

(ii) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies $\mathbf{y} = -\mathbf{x}$

(iii) $(-1)\mathbf{x} = -\mathbf{x}$

* proof is not required.

T2 Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be n elements of a vector space V.

Then $S = \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ is a subspace of V.

proof: Obviously S is a nonempty subset of V, because it contains elements of form $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$ for any $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$.

To show that S is a subspace of V, we need to show that

1) $\beta \mathbf{v} \in S$ whenever $\mathbf{v} \in S$ and $\forall \beta \in \mathbb{R}$

2) $\mathbf{v} + \mathbf{w} \in S$ whenever $\mathbf{v} \in S$ and $\mathbf{w} \in S$

proof of 1) Let $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots \alpha_n \mathbf{v}_n$ be an element of $S = \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_n)$

Let $\beta \in \mathbb{R}$ be a scalar.

To show that $\beta \mathbf{v} \in S$, we need to show that $\beta \mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_n$. But this is the case because

$$\beta \mathbf{v} = \beta(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n) = (\beta\alpha_1)\mathbf{v}_1 + \cdots + (\beta\alpha_n)\mathbf{v}_n \in S.$$

proof of 2) Let $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \in S$ and $\mathbf{w} = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n \in S$.

then $\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n + \beta_n)\mathbf{v}_n \in S$.

Hence $S = \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ is a subspace of V.

We say S is spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_n$. $\square$