Sec. 1.1, 1.2, 1.3, 1.4, 1.5

(1) Consider the system of linear equations:

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 1 \\ 3x_2 + 2x_3 - 2x_4 = 2 \\ - x_3 + 3x_4 = -2 \\ 2x_4 = 4 \end{cases}$$

a) Use back substitution to solve the system.

b) Write the augmented matrix for the system.

c) Write the system in matrix form.

(2) Write out the system of linear equations that corresponds to the following augmented matrix:

$$\left[\begin{array}{cccc|c} 4 & -2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 & 1 \\ -2 & 1 & 0 & 4 & 0 \\ 5 & 1 & 1 & -2 & 3 \end{array}\right]$$

(3) Solve the system
$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 - x_2 + x_3 = 3 \\ -x_1 + 2x_2 + 3x_3 = 7 \end{cases}$$

(4) Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(5) Determine whether the system corresponding to each of the following augmented matrices is consistent or inconsistent.

a) $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$

b) $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$

c) $\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

d) $\left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$

(6) Find the solution to the system corresponding to the following augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 5 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Determine the lead and free variables of the system.

(7) For the following systems of equations, use Gaussian elimination to obtain an equivalent system whose coefficient matrix is in row echelon form. Indicate whether the system is consistent. If the system is consistent and involves no free variables, use back substitution to find the unique solution. If the system is consistent and there are free variables, find all solutions.

a) $\begin{cases} 2x_1 + 3x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \\ 3x_1 + 4x_2 + 2x_3 = 4 \end{cases}$

b) $\begin{cases} 2x_1 - 3x_2 = 5 \\ -4x_1 + 6x_2 = 8 \end{cases}$

(8) Consider the linear system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right]$$

For what values of a will the system have a unique solution?

(9) Consider the linear system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & b & 0 \end{array}\right]$$

1) Is it possible for the system to be inconsistent? Explain.

2) For what values of b will the system have infinitely many solutions.

(10) Given a homogeneous system of linear equations, if the system is overdetermined, what are the possibilities as to the number of solutions?

(11) Given a homogeneous system of linear equations, if the system is underdetermined, what are the possibilities as to the number of solutions?

(12) Given a nonhomogeneous system of linear equations, if the system is overdetermined, what are the possibilities as to the number of solutions?

(13) Given a nonhomogeneous system of linear equations, if the system is underdetermined, what are the possibilities as to the number of solutions?

(14) If $A = \begin{bmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 7 \end{bmatrix}$ verify that $(A^T)^T = A$.

(15) If $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$ verify that $(AB)^T = B^T A^T$.

(16) If $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ 0 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, verify:

1) $(A+B)+C = A+(B+C)$

2) $(A+B)C = AC + BC$

(17) Let $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

1) Write **b** as a linear combination of the column vectors of A.

2) Write **c** as a linear combination of the column vectors of A.

(18) For each of the choices of A and **b** that follow, determine whether the system $A\mathbf{x} = \mathbf{b}$ is consistent by examining how **b** relates to the column vectors of A.

1) $A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

2) $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

(19) Explain why each of the following algebraic rules will not work in general when the real numbers a and b are replaced by $n \times n$ matrices A and B.

1) $(a+b)^2 = a^2 + 2ab + b^2$

2) $(a+b)(a-b) = a^2 - b^2$

(20) Will the rules in problem 19 work if a is replaced by $A \in \mathbb{R}^{n\times n}$ and b is replaced by the identity matrix $I \in \mathbb{R}^{n\times n}$?

(21) Let $A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Compute A^2 and A^3. What will A^n be?

(22) Let $A \in \mathbb{R}^{n \times n}$ be nonsymmetric. Show that the following matrices will be symmetric.

1) $B = A + A^T$

2) $C = A^T A$

3) $D = (I + A)(I + A^T)$

(23) Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$. Show that if $AB = A$ and $B \neq I$, then A must be singular.

(24) Let A be nonsingular. Show that A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

Hint: use the identity $(AB)^T = B^T A^T$.

(25) Let $A \in \mathbb{R}^{n \times n}$ be nonsingular. Show that A^2 is nonsingular and $(A^2)^{-1} = (A^{-1})^2$.

(26) Let $A \in \mathbb{R}^{n \times n}$. Show that if $A^2 = 0$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

(27) Given $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, show that A is nonsingular and $A^{-1} = A^T$.

(28) If $A\mathbf{x} = B\mathbf{x}$ for some nonzero vector $\mathbf{x}$, then the matrices A and B must be equal.

Hint: subtract both sides of $A\mathbf{x} = B\mathbf{x}$ from $B\mathbf{x}$.

(29) If A and B are nonsingular matrices, show that $(AB)^T$ is nonsingular and $((AB)^T)^{-1} = (A^{-1})^T (B^{-1})^T$.

(30) Which of the following matrices are elementary? Classify each elementary matrix by type.

1. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 2. $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ 3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$ 4. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(31) For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

1. $A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix}$

(32) Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}$.

1. Find an elementary matrix E_1 such that $E_1 A = B$.
2. Find an elementary matrix E_2 such that $E_2 B = C$.
3. Is C row equivalent to A? Explain.

(33) Is it possible for a singular matrix A to be row equivalent to a nonsingular matrix B? Explain.

(34) Show that if A is row equivalent to I and $AB = AC$, then $B = C$.

(35) Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$. Verify that $A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}$. Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(36) Compute the $PA = LU$ factorization of $A = \begin{pmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{pmatrix}$.

(37) Use the factorization in problem 36 to solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

Sec. 2.1, 2.2, 2.3

① Use determinants to determine whether the following matrices are nonsingular:

a) $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{bmatrix}$ d) $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{bmatrix}$

e) $\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{bmatrix}$ f) $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix}$

② Evaluate the determinant of A given below. Write your answer as a polynomial in x:

$$A = \begin{bmatrix} 1-x & 2 & 3 \\ 1 & -x & 0 \\ 0 & 1 & -x \end{bmatrix}$$

③ Find all values of λ for which $\det(A) = 0$, where $A = \begin{pmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{pmatrix}$.

④ Let $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$. Justify your answers to the following:

a) Does $\det(A+B) = \det(A) + \det(B)$?

b) Does $\det(AB) = \det(BA)$?

⑤ Consider $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix}$. Evalute $\det(A)$ by the following methods:

1) The general formula, using cofactors along the 2nd row of A.

2) The elimination method.

(6) Let $A \in \mathbb{R}^{3\times 3}$, and $\alpha \in \mathbb{R}$. Show that $\det(\alpha A) = \alpha^3 \cdot \det(A)$.

(7) Let $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 3}$ with $\det(A) = 4$ and $\det(B) = 5$.

a) Find $\det(AB)$

b) Find $\det(2A)$

c) Find $\det(A^{-1}B)$

Hint: use the following identities:

$$\det(AB) = \det(A) \cdot \det(B)$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

(8) Consider the following 3×3 elementary matrices:

E_1 : an elementary matrix of type I

E_2 : an elementary matrix of type II, formed by multiplying the second row of the identity matrix by 4

E_3 : an elementary matrix of type III

Moreover, let $A \in \mathbb{R}^{3\times 3}$ with $\det(A) = 5$.

a) Find $\det(E_1 A)$

b) Find $\det(E_1 E_2 E_3 A)$

c) Find $\det(E_1 E_2^2 A)$

(9) Let A and B be row equivalent matrices : $B = E_2 E_1 A$

where E_1 : an elementary matrix of type I

E_2 : an elementary matrix of type III

How do the values of $\det(A)$ and $\det(B)$ compare? Explain.

(10) Consider the 3×3 Vandermonde matrix:

$$V = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

a) Show that $\det(V) = (b-a)(c-a)(c-b)$.

b) what conditions must the scalars a, b, c satisfy in order for V to be nonsingular?

(11) Consider $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$.

a) Find $\operatorname{adj} A$.

b) Find $\det(A)$.

c) Find A^{-1} using parts (a) and (b).

(12) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular with $n>1$. Show that

$$\det(\operatorname{adj} A) = \left(\det(A)\right)^{n-1}.$$

Hint: use the following identities:

$$A^{-1} = \frac{\operatorname{adj} A}{\det(A)}$$

$$A^{-1}A = I$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(\alpha A) = \alpha^n \cdot \det(A), \qquad \forall \alpha \in \mathbb{R}$$

(13) Let $A \in \mathbb{R}^{4\times4}$. If $\operatorname{adj} A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{bmatrix}$.

a) Find $\det(\operatorname{adj} A)$.

b) Find $\det(A)$. Hint: use the result in problem (12).

c) Find A. Hint: use the identity $A^{-1} = \frac{\operatorname{adj} A}{\det(A)}$ and find A^{-1}. Then find $A = (A^{-1})^{-1}$.

Sec. 3.1, 3.2

① Show that $\mathbb{R}^{2\times 3}$ together with the usual addition and scalar multiplication of matrices satisfies the eight axioms of a vector space.

② Show that $C[a,b]$ together with the usual addition and scalar multiplication of functions satisfies the eight axioms of a vector space.

③ Let Z denote the set of all integers with the following two operations:

1) scalar multiplication : $\alpha \circ k = \lceil \alpha \rceil \cdot k$ for all $k \in Z$ and $\alpha \in \mathbb{R}$

where $\lceil \alpha \rceil$ is the ceil of α, which is the smallest integer greater than or equal to α. For example $\lceil 2.4 \rceil = 3$ and $\lceil 2.01 \rceil = 3$.

2) addition: $k_1 \oplus k_2 = k_1 + k_2$ (the usual addition of integers).

Which axioms of a vector space fail to hold? Motivate and justify your answer.

④ Determine whether the following sets form subspaces of $\mathbb{R}^2$:

a) $S = \{ (x_1, x_2)^T \mid x_1 + x_2 = 0 \}$

b) $S = \{ (x_1, x_2)^T \mid x_1 \cdot x_2 = 0 \}$

c) $S = \{ (x_1, x_2)^T \mid x_1 \cdot x_2 = 1 \}$

d) $S = \{ (x_1, x_2)^T \mid |x_1| = |x_2| \}$

⑤ Determine whether the following sets form subspaces of $\mathbb{R}^3$:

a) $S = \{ (x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3 \}$

b) $S = \{ (x_1, x_2, x_3)^T \mid x_1 + x_2 = 1 \}$

(6) Determine whether the following sets are subspaces of $\mathbb{R}^{2\times 2}$:

a) The set of all 2×2 diagonal matrices

b) The set of all upper triangular matrices

c) The set of all symmetric 2×2 matrices

(7) Determine whether the following sets are subspaces of P_3:

a) The set of polynomials $p(x)$ in P_3 such that $p(0) = 1$

b) The set of all linear polynomials

(8) Determine whether the following sets are subspaces of $C[-1,1]$:

a) The set of continuous nondecreasing functions on $[-1,1]$.

b) The set of continuous odd functions on $[-1,1]$.

(9) Let A be an arbitrary matrix in $\mathbb{R}^{2\times 2}$. Determine whether the following set is a subspace of $\mathbb{R}^{2\times 2}$:

$$S = \{ B \in \mathbb{R}^{2\times 2} \mid BA = 0 \}$$

(10) Determine whether the following sets are spanning sets for $\mathbb{R}^2$:

a) $S = \{ (2,1)^T, (3,2)^T \}$

b) $S = \{ (-1,2)^T, (1,-2)^T, (2,-4)^T \}$

c) $S = \{ (-1,2)^T, (1,-2)^T \}$

(11) Determine whether the following sets are spanning sets for $\mathbb{R}^3$:

a) $S = \{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$

b) $S = \{(2,1,-2)^T, (3,2,-2)^T, (4,2,-4)^T\}$

c) $S = \{(1,1,3)^T, (0,2,1)^T\}$

(12) Given $x_1 = (-1,2,3)^T$ and $x_2 = (3,4,2)^T$,

a) Is $x = (2,6,6)^T \in \text{span}(x_1, x_2)$?

b) Is $y = (-9,-2,5) \in \text{span}(x_1, x_2)$?

(13) show that the following four matrices span $\mathbb{R}^{2\times 2}$:

$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

(14) Determine whether the following sets are spanning sets for P_3:

a) $S = \{1, x, x^2\}$

b) $S = \{x+2, x^2-1\}$

c) $S = \{2, x^2, x, 2x+3\}$