Ch. 6 Eigenvalues

6.1 Eigenpairs (eigenvalues & eigenvectors)

Consider a square matrix $A \in \mathbb{R}^{n \times n}$ and a nonzero vector $\mathbf{x} \in \mathbb{R}^n$.

If $A\mathbf{x} \in \mathbb{R}^n$ is a scalar multiple of $\mathbf{x}$, that is, if

$$A\mathbf{x} = \lambda \mathbf{x}, \quad \lambda \in \mathbb{R},$$

then $\begin{cases} \lambda \text{ is called the eigenvalue of matrix } A \\ \mathbf{x} \text{ is called the eigenvector of matrix } A \end{cases}$

The pair $(\lambda, \mathbf{x})$ is called the eigenpair of matrix A.

EX. 1 Consider $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$.

Let $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $A\mathbf{x} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence, we have $A\mathbf{x} = \lambda \mathbf{x}$ with $\lambda = 2$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

This means $\begin{cases} \lambda = 2 \text{ is an eigenvalue of } A. \\ \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector of } A. \end{cases}$

Now let $\mathbf{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Then $A\mathbf{x} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Hence, we have $A\mathbf{x} = \lambda \mathbf{x}$ with $\lambda = -1$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

This means $\begin{cases} \lambda = -1 \text{ is also an eigenvector of } A. \\ \mathbf{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ is an eigenvalue of } A. \end{cases}$

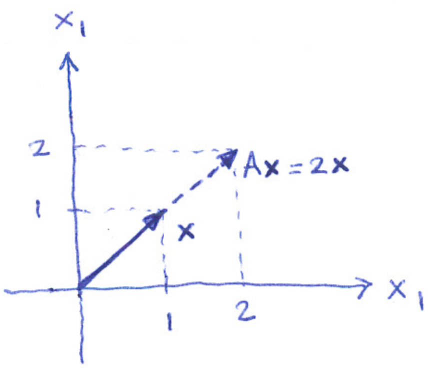
Note that a matrix can have more than one eignvalue and eigenvector.

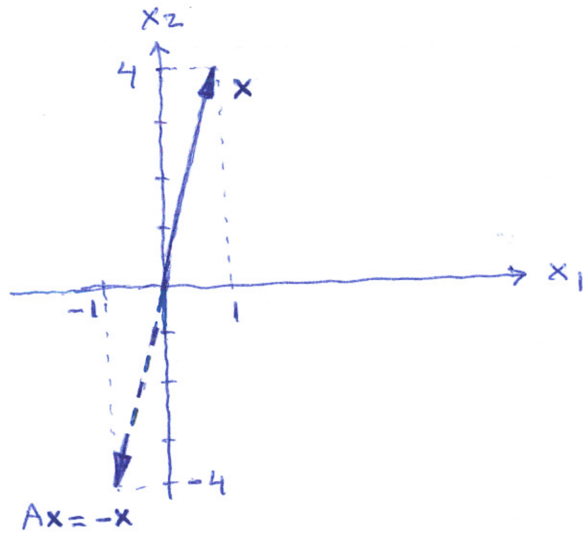
In EX.1, we say:

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector of A corresponding to the eigenvalue $\lambda = 2$,

and $x = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is the eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

Geometric representation of eigenpairs of A in EX.1:





How to compute eigenpairs of $A \in \mathbb{R}^{n \times n}$?

we need to solve the eigenvalue problem: $\boxed{A\mathbf{x} = \lambda \mathbf{x}}$

This problem can be written as:

$$(A - \lambda I)\,\mathbf{x} = 0$$

Since an eigenvector is nonzero, λ will be an eigenvalue of A iff the above homogeneous system has a nontrivial (nonzero) solution $\mathbf{x} \neq 0$.

This will be the case iff the coefficient matrix $A - \lambda I$ is singular, that is, if $\det(A - \lambda I) = 0$.

The equation $\boxed{\det(A-\lambda I)=0}$ is called the characteristic equation.

In fact $\det(A-\lambda I)$ is an nth-degree polynomial $p(\lambda)$ in variable λ, and hence the characteristic equation has n solutions. By solving the characteristic equation for λ, we obtain n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Note that the eigenvalues $\lambda_1, \cdots, \lambda_n$ are the n roots of the nth-degree polynomial $p(\lambda) = \det(A-\lambda I)$. The polynomial $p(\lambda)$ may have multiple roots. In this case some eigenvalues will be repeated. The polynomial $p(\lambda)$ may also have complex roots. In this case some eigenvalues will be complex-valued.

In any case, a matrix $A \in \mathbb{R}^{n \times n}$ has n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, some of which may be repeated and/or some of which may be complex numbers.

How to find eigenvectors?

After finding an eigenvalue, say λ_i, we will find its corresponding eigenvector, say $\mathbf{x}^{(i)}$, by solving the following system:

$$\boxed{(A-\lambda_i I)\,\mathbf{x}^{(i)} = \mathbf{0}}$$

EX. 2 Find the eigenpairs of $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$.

STEP 1 $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(-2-\lambda) + 4 = 0 \Rightarrow \boxed{\lambda^2 - \lambda - 2 = 0}$$

$$\Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = -1 \end{cases}$$

STEP 2 $(A - \lambda I)\mathbf{x} = \mathbf{0}$

For $\lambda_1 = 2$: $\begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 - x_2 = 0 \\ 4x_1 - 4x_2 = 0 \end{cases} \Rightarrow x_1 = x_2$

$$\Rightarrow \mathbf{x}^{(1)} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}, \ \alpha \neq 0.$$

Any nonzero multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

For $\lambda_2 = -1$: $\begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4x_1 - x_2 = 0 \\ 4x_1 - x_2 = 0 \end{cases} \Rightarrow 4x_1 = x_2$

$$\Rightarrow \mathbf{x}^{(2)} = \beta \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \beta \in \mathbb{R}, \ \beta \neq 0.$$

Any nonzero multiple of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$.

Complex eigenvalues

Let $A \in \mathbb{R}^{n\times n}$. If the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has complex roots, they will be conjugate. This means that if $\lambda = a + bi$ $(b \neq 0)$ is an eigenvalue, then $\lambda^* = a - bi$ will also be an eigenvalue. Moreover, the eigenvectors corresponding to λ and λ^* will also be conjugate. This means if $\mathbf{x}$ is the eigenvector corresponding to λ, then $\mathbf{x}^*$ will be the eigenvectors corresponding to the eigenvalue λ^*.

EX. 3 Find the eigenpairs of $A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$.

STEP 1: $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -\frac{1}{2} - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \left(\tfrac{1}{2} + \lambda\right)^2 + 1 = 0 \Rightarrow \tfrac{1}{2} + \lambda = \pm i \Rightarrow \lambda = \tfrac{-1}{2} \pm i \Rightarrow \begin{cases} \lambda_1 = \frac{-1}{2} + i \\ \lambda_2 = \frac{-1}{2} - i \end{cases}$$

We observe that λ_1 and λ_2 are complex conjugate: $\lambda_2 = \lambda_1^*$.

It is therefore enough to find only the eigenvector corresponding to λ_1

STEP 2: $(A - \lambda I)\mathbf{x} = \mathbf{0}$

For $\lambda_1 = \frac{-1}{2} + i$:
$$\begin{pmatrix} \frac{-1}{2} + \frac{1}{2} - i & 1 \\ -1 & \frac{-1}{2} + \frac{1}{2} - i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -i x_1 + x_2 = 0 \\ -x_1 - i x_2 = 0 \end{cases}$$

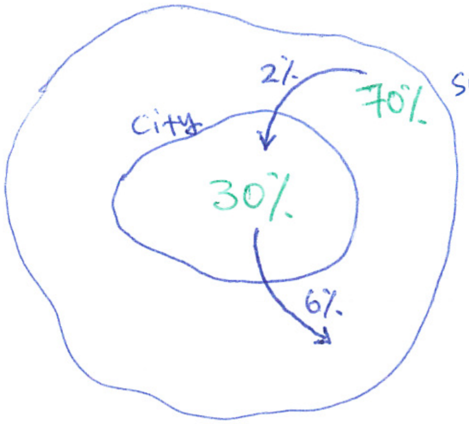
$$\Rightarrow i x_1 = x_2 \quad \Rightarrow \quad \mathbf{x}^{(1)} = \alpha \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \alpha \in \mathbb{R},\ \alpha \neq 0.$$

It follows that $\mathbf{x}^{(2)} = \alpha \begin{pmatrix} 1 \\ -i \end{pmatrix}$, which is the conjugate of $\mathbf{x}^{(1)}$.

As an exercise find $\mathbf{x}^{(2)}$ by solving $(A - \lambda_2 I)\,\mathbf{x}^{(2)} = \mathbf{0}$ and verify that $\mathbf{x}^{(2)} = \mathbf{x}^{(1)*}$.

Application of eigenpairs in changing basis

Recall population migration problem; see lecture notes on chapter 3, page 20.



$$\mathbf{x}_n = \begin{pmatrix} \text{population\% in city in } n \text{ years} \\ \text{population\% in suburbs in } n \text{ years} \end{pmatrix}$$

$$A = \begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$$

$$x_1 = A x_0 = \begin{pmatrix} 0.94 \times 0.3 + 0.02 \times 0.7 \\ 0.06 \times 0.3 + 0.98 \times 0.7 \end{pmatrix}$$

$$x_2 = A^2 x_0$$

$$x_3 = A^3 x_0$$

$$\vdots$$

$x_n = A^n x_0$ difficult to compute/study x_n when $n \to \infty$.

Let us now change the basis to $S = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$ where $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are the eigenvectors of A.

If we compute eigenpairs of A, we get:
$$\begin{cases} \lambda_1 = 1, \ \mathbf{x}^{(1)} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow A\mathbf{x}^{(1)} = \mathbf{x}^{(1)} \\ \lambda_2 = 0.92, \ \mathbf{x}^{(2)} = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow A\mathbf{x}^{(2)} = 0.92\,\mathbf{x}^{(2)} \end{cases}$$

We then write $\mathbf{x}_0$ in the new basis: $\mathbf{x}_0 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} = 0.25\,\mathbf{x}^{(1)} - 0.05\,\mathbf{x}^{(2)}$.

then $\mathbf{x}_n = A^n \mathbf{x}_0 = 0.25 \underbrace{A^n \mathbf{x}^{(1)}}_{\mathbf{x}^{(1)}} - 0.05\, A^n \mathbf{x}^{(2)} = 0.25\, \mathbf{x}^{(1)} - 0.05\,(0.92)^n\, \mathbf{x}^{(2)}$

Hence, $\lim_{n \to \infty} x_n = 0.25\,\mathbf{x}^{(1)} = 0.25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$