Lecture Notes, Math 314, Spring 2017, M. MOTAMED

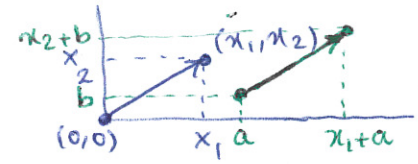
ch.3 Vector Spaces

3.1 Definition

① Euclidean Vector Space $\mathbb{R}^n$ (the most basic, elementary VS)

$\mathbb{R}^2$: a vector in $\mathbb{R}^2$ can be geometrically represented by a directed line segment.

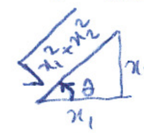
A non-zero vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is represented in the plane by a directed line segment from $(0,0)$ to (x_1, x_2)



At the same time $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ can be represented by any line segment that has the same length and direction: e.g. from (a,b) to (x_1+a, x_2+b).

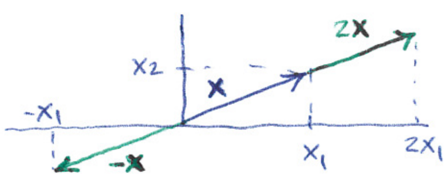
they uniquely determine x_1, x_2

1) Euclidean length of a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}^2$ is the length of any directed line segment representing $\mathbf{x}$: $\sqrt{x_1^2 + x_2^2}$



2) direction: $\sin(\theta) = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$

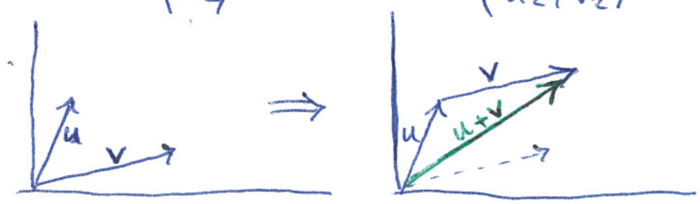
3) Multiplication of $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ by a scalar α : $\alpha \mathbf{x} = \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$



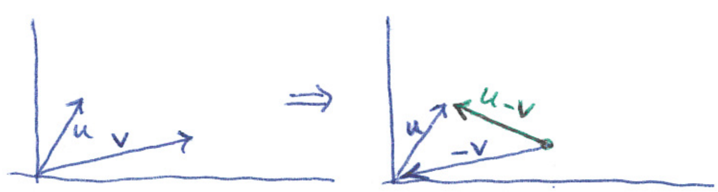
$\alpha = 2$: in the same direction, with a length 2 times that of $\mathbf{x}$

$\alpha = -1$: in the opposite direction, with the same length

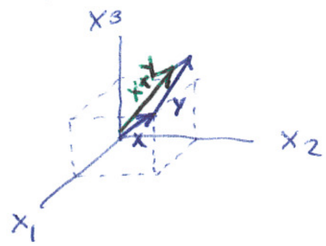
4) Sum of two vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$: $\mathbf{u}+\mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$



what about $\mathbf{u}-\mathbf{v}$?



$\mathbb{R}^3$: a vector in $\mathbb{R}^3$ $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ can be represented by directed line segments in a 3-dimensional space.



$\mathbb{R}^n$: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ scalar multiplication: $\alpha \mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$ addition: $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$ $\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

* We can view $\mathbb{R}^n$ as the set of all $n \times 1$ matrices with real entries. The addition & scalar multiplication of vectors in $\mathbb{R}^n$ is just the usual $+$ & $\times$ of matrices.

② The vector space $\mathbb{R}^{m \times n}$

$\mathbb{R}^{m \times n}$ is the set of all $m \times n$ matrices with real entries.

if $A = (a_{ij})$ and $B = (b_{ij})$ then the sum $A + B = (a_{ij} + b_{ij})$

Given a scalar α, αA is defined as a matrix with entries (αa_{ij})

By defining each operation, we create/set a rule in the space.
This way we can create a mathematical system.
all members or elements of the space (here matrices) obey the rules, (called axioms)

Vector Space and its Axioms

Let V be a set on which the operations of addition & scalar multiplication are defined.
This means for each pair of elements $\mathbf{x}$ and $\mathbf{y}$ in V, and for each scalar α

$\mathbf{x} + \mathbf{y}$ is defined (the result is a unique element)

$\alpha \mathbf{x}$ is defined (the result is a unique element)

with the following closure properties:
$\begin{cases} C_1: & \text{if } \mathbf{x} \in V \text{ and } \alpha \text{ is a scalar, then } \alpha\mathbf{x} \in V \\ C_2: & \text{if } \mathbf{x}, \mathbf{y} \in V, \text{ then } \mathbf{x} + \mathbf{y} \in V \end{cases}$

The set V together with the operations of addition & scalar multiplication is said to form a vector space if the following axioms are satisfied:

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ $\forall\, \mathbf{x}, \mathbf{y} \in V$

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ $\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$

A3. $\exists$ an element $\mathbf{0} \in V$ s.t. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ $\forall\, \mathbf{x} \in V$

A4. $\forall\, \mathbf{x} \in V$, $\exists$ an element $-\mathbf{x} \in V$ s.t. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

A5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y}$ $\forall$ scalar α & $\forall\, \mathbf{x}, \mathbf{y} \in V$

A6. $(\alpha + \beta)\mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{y}$

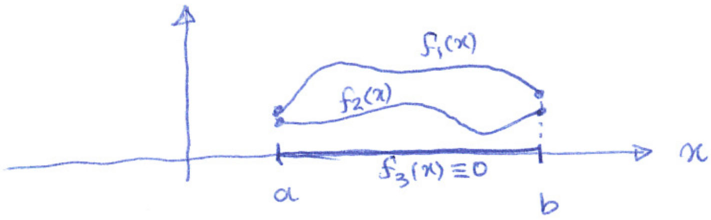
A7. $(\alpha\beta)\mathbf{x} = \alpha(\beta \mathbf{x})$

A8. $1\mathbf{x} = \mathbf{x}$ $\forall\, \mathbf{x} \in V$

V is called the universal set for the vector space. Its elements are called vectors.

③ The vector space $C[a,b]$

$C[a,b]$ denotes the set of all real-valued functions defined on the closed interval $[a,b]$ that are continuous on $[a,b]$.



In this case are vectors are functions, and our universal set V is $C[a,b]$ (the set of cont. fcn's on $[a,b]$)

Let us for example verify the closure property C_2 in this case:

- The sum of two fcn's in $C[a,b]$ is defined by $(f+g)(x) = f(x) + g(x)$ $\forall x \in [a,b]$
- the new fcn $f+g$ is an element of $C[a,b]$, because the sum of two cont. fcn's is cont.

Let us verify the 1st closure property C_1:

- αf is defined by $(\alpha f)(x) = \alpha \cdot f(x)$ $\forall \alpha \in \mathbb{R}$, $\forall x \in [a,b]$
- The new fcn αf is an element of $C[a,b]$, because a constant times a cont. fcn is always cont.

Thus we have defined the operations of <u>addition</u> & <u>scalar multiplication</u> on $C[a,b]$ and verified their closure properties.

It is easy to show that the axioms also hold:

For ex. A1: $(f+g)(x) = (g+f)(x)$ $\forall x \in [a,b]$

because: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$ $\forall x \in [a,b]$.

<u>Note</u>: The zero vector here will be $\boxed{z(x) = 0 \quad \forall x \in [a,b]}$

$\Downarrow$

notation $\underline{z(x) \equiv 0}$

(4) The vector space P_n

P_n denotes the set of all polynomials of degree less than n.

A general polynomial of degree at most $n-1$ has the form:

$$p(x) = a_n \cdot x^{n-1} + a_{n-1} \cdot x^{n-2} + \cdots + a_2 \cdot x + a_1 \qquad a_i \in \mathbb{R},\ i=1,2,\dots,n$$

(ex) $n=3$: $\quad p(x) = a_3 x^2 + a_2 x + a_1 \qquad a_1, a_2, a_3 \in \mathbb{R}$

addition: $(p+q)(x) = p(x) + q(x)$

scalar multip.: $(\alpha p)(x) = \alpha \cdot p(x)$

zero vector: the zero polynomial $\quad Z(x) = 0 \cdot x^{n-1} + 0 \cdot x^{n-2} + \cdots + 0 \cdot x + 0$

↳ a 0-degree polynomial

Axioms can be verified easily.

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three more fundamental properties of vector spaces:

Theorem If V is a vector space and $\mathbf{x}$ is any element of V, then

(i) $0\mathbf{x} = \mathbf{0}$

(ii) $\mathbf{x}+\mathbf{y} = \mathbf{0}$ implies $\mathbf{y} = -\mathbf{x}$ (additive inverse of $\mathbf{x}$ is unique)

(iii) $(-1)\mathbf{x} = -\mathbf{x}$

proof: (i) A8 $\Rightarrow$ $\mathbf{x} = 1\mathbf{x} = (1+0)\mathbf{x} \overset{A6}{=} 1\mathbf{x} + 0\mathbf{x} \overset{A8}{=} \mathbf{x} + 0\mathbf{x} \overset{A3}{\Longrightarrow} 0\mathbf{x} = \mathbf{0}$

or $\underbrace{-\mathbf{x}+\mathbf{x}}_{0 \text{ by } A4} = -\mathbf{x} + (\mathbf{x} + 0\mathbf{x}) \overset{A2}{=} \underbrace{(-\mathbf{x}+\mathbf{x})}_{0 \text{ by } A4} + 0\mathbf{x} = \mathbf{0} + 0\mathbf{x} = 0\mathbf{x}$

We can define a one-to-one correspondence between the elements of P_n and $\mathbb{R}^n$ by $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1} \longleftrightarrow \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Then, we can show if $p \leftrightarrow \mathbf{a}$ and $q \leftrightarrow \mathbf{b}$, then $\begin{cases} \alpha p \leftrightarrow \alpha \mathbf{a} & \forall \text{ scalar } \alpha \\ p+q \leftrightarrow \mathbf{a}+\mathbf{b} \end{cases}$

Two vector spaces are said to be isomorphic if their elements can be put into a one-to-one correspondence that is preserved under scalar multiplication & addition.

3.2 Subspaces

Consider a vector space V with operations $\begin{cases} \text{addition} \\ \text{scalar mult.} \end{cases}$

Can we create another vectors space by taking a subset S of V and using the operations of V?

For this we need S to be closed under the operations:

1) the sum of two elements of S must always be an element of S
2) the product of a scalar and an element of S must always be an el. of S.

EX.1 Let $V = \mathbb{R}^2$. The elements of V are vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

We can write: $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, x_1 \in \mathbb{R}, x_2 \in \mathbb{R} \right\}$

Now let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \; x_2 = 2x_1 \right\}$ or $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_2 = 2x_1 \right\}$

Is S a subspace of V?

Take an element of S: $x = \begin{pmatrix} a \\ 2a \end{pmatrix}$.

1) $\alpha x = \begin{pmatrix} \alpha a \\ \alpha 2a \end{pmatrix} = \begin{pmatrix} \alpha a \\ 2(\alpha a) \end{pmatrix} \in S$

2) $x + y = \begin{pmatrix} a \\ 2a \end{pmatrix} + \begin{pmatrix} b \\ 2b \end{pmatrix} = \begin{pmatrix} a+b \\ 2(a+b) \end{pmatrix} \in S$

$\Rightarrow$ S is a subset of V and is itself a vector space

$\Rightarrow$ S is a subspace of $V = \mathbb{R}^2$.

Def. If S is a <u>nonempty</u> <u>subset</u> of a vector space V satisfying
1 2

3 (i) $\alpha x \in S$ whenever $x \in S$ and $\forall$ scalar α

4 (ii) $x + y \in S$ whenever $x \in S$ and $y \in S$

then S is said to be a <u>subspace</u> of V.

Equivalently, a subspace of V is a subset of V that is closed under the operations of V.

Clearly, since a subspace with two operations inherited from a vector space satisfy the axioms of a vector space, every <u>subspace</u> of a vector space is a <u>vector space</u>.

<u>Remark</u>. obviously $\{0\}$ is always a subspace of V.
It is called the zero subspace.
Note that $\{0\}$ is nonempty and is closed under $+$ & scalar $\times$.

In order to show that a subset S of a vector space V is a subspace, we need to show:
1) S is nonempty $\xrightarrow[\text{(this is equivalent to)}]{\text{Since every subspace must contain } 0}$ need to show $0 \in S$
2) S is closed under $+$ & sc. $\times$

<u>EX. 2</u> Let $V = \mathbb{R}^3$. Let $S = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = x_2\}$.

1) S is nonempty since $0 = (0,0,0)^T \in S$.

2) closure properties: (i) For any vector $x = \begin{pmatrix} a \\ a \\ b \end{pmatrix}$ in S, $\alpha x = \begin{pmatrix} \alpha a \\ \alpha a \\ \alpha b \end{pmatrix} \in S$

(ii) If $x = \begin{pmatrix} a \\ a \\ b \end{pmatrix}$ and $y = \begin{pmatrix} c \\ c \\ d \end{pmatrix}$ are two arbitrary elements of S

then $x + y = \begin{pmatrix} a \\ a \\ b \end{pmatrix} + \begin{pmatrix} c \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ a+c \\ b+d \end{pmatrix} \in S$.

Hence, S is a subspace of V.

<u>EX. 3</u> Let $V = \mathbb{R}^{2\times 2}$. Let $S = \{ A = (a_{ij}) \in \mathbb{R}^{2\times 2} \mid a_{21} = 0 \}$.

1) S is nonempty since $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$.

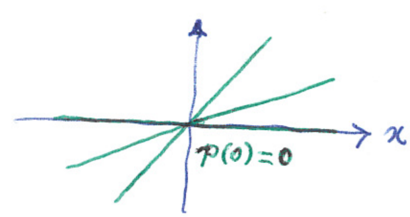
2) closure: (i) $\forall\, A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$, $\alpha A = \begin{pmatrix} \alpha a & \alpha b \\ 0 & \alpha c \end{pmatrix} \in S$.

(ii) $\forall\, A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$, $\forall\, B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \in S$, $A+B = \begin{pmatrix} a+d & b+e \\ 0 & c+f \end{pmatrix} \in S$

Hence, S is a subspace of V.

<u>EX. 4</u> Let $V = P_2$: all polynomials of degree at most 1.

Let $S = \{ p(x) \in P_2 \mid p(0) = 0 \}$



1) S is nonempty since it contains the zero polynomial.

2) (i) if $p(x) \in S$ and $\alpha \in \mathbb{R}$, then $\alpha p(0) = \alpha \cdot 0 = 0 \Rightarrow \alpha p \in S$

(ii) if $p(x), q(x) \in S$, then $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0 \Rightarrow p+q \in S$.

Hence, S is a subspace of V.

⑤ The vector space $C^n[a,b]$

$C^n[a,b]$ denotes the set of all real-valued functions f defined on $[a,b]$ so that $f, f', f'', \dots, f^{(n)}$ are continuous on $[a,b]$.

C^n is called the space of n-times continuously differentible functions.

Remark. $C^n[a,b]$ is a subspace of $C[a,b]$ (every n-times cont. diff fcn is a continuous function)

P_n is a subspace $C[a,b]$. (every polynomial of degree at most n-1 is a continuous fcn.)

$C[a,b]$ is not a subspace of $C^n[a,b]$ for $n \geq 1$.

For ex. $f(x) = |x| \in C[-1,1]$ but $\notin C^1[-1,1] \Rightarrow C[-1,1]$ is not a subset of $C^1[-1,1]$.

$\boxed{\text{NOTE}}$ S must be a subset of V.

This means that each element of S must be in V.

The span of a set of vectors

Recall: If $v_1, v_2, \dots, v_3$ are vectors and $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars, the sum of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is called a linear combination of the vectors $v_1, v_2, \dots, v_n$.

Def. Let $v_1, \dots, v_n$ be n vectors in a vector space V.
The set of all linear combinations of $v_1, \dots, v_n$ is called the span of $v_1, \dots, v_n$ and is denoted by $\text{span}(v_1, \dots, v_n)$.

EX. 5 Let $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

For any scalar $\alpha, \beta \in \mathbb{R}$, we can obtain a linear comb. of v_1 and v_2:

$$\alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha - \beta \\ 2\alpha \end{pmatrix}$$

This is a vector.
The set of all such vectors, obtained from different values of α and β is the $\text{span}(v_1, v_2)$, which can be written as:

$$\text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) = \left\{ v = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R}, \beta \in \mathbb{R} \right\}$$

for example, this set contains the following vectors:

$\alpha = \beta = 0 \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\alpha = \beta = 1 \Rightarrow \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\alpha = 1, \beta = 0 \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\alpha = 0, \beta = -1 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\alpha = \frac{1}{2}, \beta = \frac{-1}{2} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

It contains infinitely many vectors, corresponding to infinitely many choices of $\alpha, \beta \in \mathbb{R}$.

Ex. 6 Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

what is the span(e_1, e_2, e_3) ?

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad \text{where } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

Hence $\text{span}(e_1, e_2, e_3) = \mathbb{R}^3$.

Ex. 7 Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

what is the span(e_1, e_2) ?

$$\alpha_1 e_1 + \alpha_2 e_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \quad \text{where } \alpha_1, \alpha_2 \in \mathbb{R}$$

The third entry is always zero. Therefore $\text{span}(e_1, e_2) \neq \mathbb{R}^3$

But, we can show that span(e_1, e_2) is a subspace of $\mathbb{R}^3$.

1) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \text{span}(e_1, e_2)$

2) if $\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in \text{span}(e_1, e_2)$ and $\beta \in \mathbb{R} \Rightarrow \beta \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta x_1 \\ \beta x_2 \\ 0 \end{pmatrix} \in \text{Span}(e_1, e_2)$

3) if $\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} \in \text{span}(e_1, e_2) \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 0 \end{pmatrix} \in \text{span}(e_1, e_2)$

Theorem Let $v_1, \dots, v_n$ be elements of a vector space V. Then $\text{span}(v_1, \dots, v_n)$ is a subspace of V.

proof. 1) Let $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ be an element of $\text{span}(v_1, \dots, v_n)$. Let β be a scalar.

we show that βv is also an element of $\text{span}(v_1, \dots, v_n)$. Equivalently, we show that βv can be written as a linear comb. of v_i's:

$$\beta v = \beta(\alpha_1 v_1 + \dots + \alpha_n v_n) = (\beta\alpha_1) v_1 + \dots + (\beta\alpha_n) v_n \quad \checkmark$$

2) Let $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ and $w = \beta_1 v_1 + \dots + \beta_n v_n \in \text{span}(v_1, \dots, v_n)$

then $v + w = (\alpha_1 + \beta_1) v_1 + \dots + (\alpha_n + \beta_n) v_n \in \text{Span}(v_1, \dots, v_n)$ ✓

We say $\text{span}(v_1, \dots, v_n)$ is a subspace of V spanned by $v_1, \dots, v_n$. □

Spanning set for a vector space

We saw that $\text{span}(v_1, \cdots, v_n) \overset{\text{subspace}}{\subset} V$.

It may happen that $\text{span}(v_1, \cdots, v_n) = V$.
In this case we say vectors $v_1, \cdots, v_n$ span V, or $\{v_1, \cdots, v_n\}$ is a <u>spanning set</u> for V.

<u>Def.</u> The set $\{v_1, \cdots, v_n\}$ is a spanning set for V iff every vector in V can be written as a linear combination of $v_1, \cdots, v_n$.

<u>see Ex.6</u>

<u>Ex.8</u> Which of the following are spanning sets of $\mathbb{R}^3$?

a) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$

b) $\{(1,0,1)^T, (0,1,0)^T\}$

To determine whether a set of vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$, we need to consider a $\underset{\text{arbitrary}}{\underline{\text{general}}}$ vector $(a,b,c)^T$ in $\mathbb{R}^3$ and investigate if it can be written a linear combination of the vectors in the set.

a) need to determine if it is possible to find constants $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = a \\ \alpha_1 + \alpha_2 = b \\ \alpha_1 = c \end{cases}$$

or

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\det(A) = -1 \neq 0 \Longrightarrow \exists$ unique sol. $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix}$

$\Longrightarrow$ The three vectors span $\mathbb{R}^3$.

b)

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix} \Longrightarrow$$ It only produces vectors whose 1st & 3rd entries are equal.

$\Rightarrow \text{span}((1,0,1)^T, (0,1,0)^T)$ is only a subspace of $\mathbb{R}^3$

<u>Ex.9</u> Vectors $1-x^2$, $x+2$, x^2 span P_3. Thus any polynomial ax^2+bx+c in P_3 can be written as a linear comb. of these three vectors. $\Rightarrow$ It is possible to find $\alpha_1, \alpha_2, \alpha_3$ s.t.

$\alpha_1(1-x^2) + \alpha_2(x+2) + \alpha_3 x^2 = ax^2 + bx + c \Longrightarrow \begin{cases} -\alpha_1 + \alpha_3 = a \\ \alpha_2 = b \\ \alpha_1 + 2\alpha_2 = c \end{cases} \Rightarrow \boxed{\begin{matrix} \alpha_1 = c-2b \\ \alpha_2 = b \\ \alpha_3 = a+c-2b \end{matrix}}$

$\det \begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -1 \neq 0$

3.3 linear independence

Consider a spanning set that spans a vector space.

Recall Consider the following sets:

$$S_1 = \{ (1,0)^T, (0,1)^T, (1,1)^T \}$$

$$S_2 = \{ (1,0)^T, (0,1)^T \}$$

We know that both S_1 and S_2 span $\mathbb{R}^2$.

We can show this by considering a general vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$.

$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$	$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$
α_3 is the free variable	$\Downarrow \det(A) \neq 0$
$\Rightarrow \exists$ many solutions $(\alpha_1, \alpha_2, \alpha_3)$	$\exists$ unique (α_1, α_2)
we can pick $\alpha_3 = 0$	
this means $(1,1)^T$ in not needed to span $\mathbb{R}^2$.	

It follows that $\text{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \text{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \mathbb{R}^2$

In practice, it is desirable to find a minimal spanning set, i.e., a spanning set with no unnecessary elements.

For this, we need to investigate how the vectors in the collection depend on each other.

$\Rightarrow$ we need to study the concepts of $\begin{cases} \text{linear dependence} \\ \text{linear independence} \end{cases}$

Def. The vectors $v_1, \ldots, v_n$ in a vector space V are said to be linearly independent if $c_1 v_1 + \cdots + c_n v_n = 0$ implies that all the scalars $c_1, \ldots, c_n$ must equal 0.

We have the following results:

1) if $\{v_1, \ldots, v_n\}$ is a minimal spanning set $\Rightarrow$ $v_1, \ldots, v_n$ are linearly indep.

2) if $\begin{cases} v_1, \ldots, v_n \text{ span } V \\ \text{and} \\ v_1, \ldots, v_n \text{ are linearly indep.} \end{cases}$ $\Rightarrow$ $\{v_1, \ldots, v_n\}$ is a minimal spanning set for V

Ex. $S_1 = \{ (1,0)^T, (0,1)^T, (1,1)^T \}$ spans $\mathbb{R}^2$

But $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\end{pmatrix}$ are not linearly independent

because $c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} + c_3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \not\Rightarrow c_1 = c_2 = c_3 = 0$
the system has many non-zero solutions, for ex. $c_1 = 1, c_2 = 1, c_3 = -1$ is a solution

Hence, $S_1 = \{v_1, v_2, v_3\}$ is not a minimal spanning set for $\mathbb{R}^2$.

Ex. $S_2 = \{ (1,0)^T, (0,1)^T \}$ spans $\mathbb{R}^2$

$v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $v_2 \begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent

because $c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$
$\Rightarrow \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow c_1 = c_2 = 0$

Hence, $S_2 = \{v_1, v_2\}$ is a minimal spanning set for $\mathbb{R}^2$.

Def. the vectors $v_1, \ldots, v_n$ in a vector space V are said to be linearly dependent if there exist scalars $c_1, \ldots, c_n$, not all zero, s.t. $c_1 v_1 + \cdots + c_n v_n = 0$.

Ex. $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\end{pmatrix}$ are linearly independent, because

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ holds when for example $c_1 = 1, c_2 = 1, c_3 = -1$.

theorem. Let $x_1, \cdots, x_n \in \mathbb{R}^{n\times 1}$. Consider the square matrix $X = (x_1, \cdots, x_n)$.

The vectors $x_1, \cdots, x_n$ are linearly dependent iff $\det(X) = 0$.

proof. The equation $c_1 x_1 + \cdots + c_n x_n = 0$ can be written as

$$Xc = 0 \quad \text{where } c = (c_1, \cdots, c_n)^T.$$

This system has a non-trivial solution (a non-zero sol.) iff $\det(X) = 0$.

Ex. Determine whether the vectors $\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}$ are linearly dep.

$$\det\begin{pmatrix}1 & -2 & 0\\ 2 & 1 & 1\\ 0 & 1 & 3\end{pmatrix} = 1\times 2 + 2\times 6 = 15 \neq 0 \Rightarrow \text{they are linearly indep.}$$

What if we have $x_1, \cdots, x_k \in \mathbb{R}^{n\times 1}$ where $k \neq n$?

Then matrix $X = (x_1, \cdots, x_k) \in \mathbb{R}^{n\times k}$ is not a square matrix, and we cannot use determinant of X to determine whether the vectors are linearly dep.

Ex. Determine whether $x_1 = \begin{pmatrix}1\\-1\\2\\3\end{pmatrix}$, $x_2 = \begin{pmatrix}-2\\3\\1\\-2\end{pmatrix}$, $x_3 = \begin{pmatrix}1\\0\\7\\7\end{pmatrix}$ are linearly dependent.

Form the system $\underbrace{Xc = 0}_{\substack{\text{overdetermined}\\ \text{homogeneous}\\ \text{system}}} \Rightarrow \left(\begin{array}{ccc|c}1 & -2 & 1 & 0\\ -1 & 3 & 0 & 0\\ 2 & 1 & 7 & 0\\ 3 & -2 & 7 & 0\end{array}\right) \xrightarrow{\text{elimination}} \underbrace{\left(\begin{array}{ccc|c}1 & -2 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)}_{\text{RE form}}$

Since $\exists$ a free variable $\Rightarrow \exists$ many solutions $c = (c_1, c_2, c_3)^T$, not only one solution $c_1 = c_2 = c_3 = 0$. $\Rightarrow$ the vectors are linearly dependent.

What if $Xc = 0$ is underdetermined? Then $\exists$ always atleast one free variables. Hence, $\exists$ always many nonzero solutions $\Rightarrow$ always linearly dependent.

Theorem Consider $v_1, \cdots, v_n$ in a vector space V. Let $S = \text{span}(v_1, \cdots, v_n)$.

A vector $v \in S$ can be written uniquely as a linear combination of $v_1, \cdots, v_n$ iff $v_1, \cdots, v_n$ are linearly independent.

Vector spaces of functions

(P_n) Consider k polynomials $p_1, p_2, \dots, p_k \in P_n$.

set $\underbrace{c_1 p_1(x) + \dots + c_k p_k(x)}_{\text{rewrite this in the form } a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n} = \underbrace{Z(x)}_{\text{zero polynomial}}$

$Z(x) = 0 \cdot x^{n-1} + 0 \cdot x^{n-2} + \dots + 0x + 0$

then equate the corresponding coeff's: $\begin{cases} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_n = 0 \end{cases} \Rightarrow$ $\boxed{\text{obtain a homogeneous linear system for } c_1, \dots, c_k}$.

Ex. Determine whether the polynomials $\begin{cases} p_1(x) = x^2 - 2x + 3 \\ p_2(x) = 2x^2 + x + 8 \\ p_3(x) = x^2 + 8x + 7 \end{cases}$ are linearly dependent.

set $c_1(x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7) = 0x^2 + 0x + 0$

$\Rightarrow (c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0x^2 + 0x + 0$

$\Rightarrow \begin{cases} c_1 + 2c_2 + c_3 = 0 \\ -2c_1 + c_2 + 8c_3 = 0 \\ 3c_1 + 8c_2 + 7c_3 = 0 \end{cases} \Rightarrow$ Coeff. matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{pmatrix} \Rightarrow \det(A) = 0$

$\Downarrow$

the polynomials are linearly dep.

---

what about functions?

Consider n functions: $f_1, f_2, \dots, f_n$

Set $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$

Need $n-1$ more equations. We obtain them by taking derivatives of

$$\Longrightarrow \quad c_1 f_1'(x) + c_2 f_2'(x) + \dots + c_n f_n'(x) = 0$$

$$c_1 f_1''(x) + c_2 f_2''(x) + \dots + c_n f_n''(x) = 0$$

$$\vdots$$

$$c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \dots + c_n f_n''(x) = 0$$

$\Rightarrow$ we get the system

$$\begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We need to check determinant of the coefficient matrix, which is called the Wronskian of $f_1, \dots, f_n$:

$$W[f_1, \dots, f_n](x) = \det \begin{pmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & & f_n^{(n-1)}(x) \end{pmatrix}.$$

Note for $(n-1)$ derivatives of $f_1, \dots, f_n$, we need $f_1, \dots, f_n \in C^{n-1}([a,b])$

Theorem Let $f_1, \dots, f_n \in C^{n-1}[a,b]$. If $\exists$ a point $x_0 \in [a,b]$ s.t. $W[f_1, \dots, f_n](x_0) \neq 0$, then $f_1, \dots, f_n$ are linearly independent.

EX. Show that e^x and e^{-x} are linearly independent over $(-\infty, \infty)$.

$$W[e^x, e^{-x}] = \det \begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix} = -e^{x-x} - e^{x-x} = -2 \neq 0 \Rightarrow e^x \,\&\, e^{-x} \text{ are linearly indep.}$$

What if $W = 0$? The converse of the above theorem is not valid.

If $f_1, \dots, f_n$ are linearly dependent $\Rightarrow W[f_1, \dots, f_n](x) = 0 \;\; \forall x \in [a,b]$

$\not\Leftarrow$

NOTE: this holds for ODE's

EX. let $\begin{cases} f_1(x) = x^2 \\ f_2(x) = x|x| \end{cases}$ on $[-1, 1]$.

$$W[f_1, f_2](x) = \det \begin{pmatrix} x^2 & x|x| \\ 2x & 2|x| \end{pmatrix} = 2|x|x^2 - 2x^2|x| = 0 \Rightarrow$$

We cannot conclude from this any thing!

set $c_1 x^2 + c_2 x|x| = 0 \quad \forall x \in [-1, 1]$

think of $f_i(x)$ as a vector in $\mathbb{R}^2$: $f_1 = \begin{pmatrix} f_1(+1) \\ f_1(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $f_2 = \begin{pmatrix} f_2(1) \\ f_2(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow$ check if these two vectors are dep. or indep.

$$\begin{cases} x = 1 \Rightarrow c_1 + c_2 = 0 \\ x = -1 \Rightarrow c_1 - c_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0$$

$\Rightarrow$ the only sol. is $c_1 = c_2 = 0 \Rightarrow f_1 \,\&\, f_2$ are linearly indep. even though $W[f_1, f_2] = 0$

NOTE: $f_1(x), \dots, f_n(x)$ are linearly dep. if $\exists$ non-zero $c_1, \dots, c_n$ s.t. $c_1 \cdot f_1(x) + \cdots + c_n \cdot f_n(x) = 0 \quad \forall x \in [a,b]$ otherwise lin. indep.

3.4 Basis & Dimension

In sec. 3.4 we showed that a spanning set for a vector space is minimal if its elements are linearly indep.

The elements of a minimal spanning set form the basic building blocks for the whole vector space. We say they form a basis for the vec. space.

Def. The vectors. $v_1, \dots, v_n$ form a basis for a vector space V iff

(i) $v_1, \dots, v_n$ are linearly indep.

(ii) $v_1, \dots, v_n$ span V.

Ex. Let $S = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ where $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Let $V = \mathbb{R}^{2\times 2}$.

(i) are $E_{11}, E_{12}, E_{21}, E_{22}$ linearly indep. ?

$$c_1 E_{11} + c_2 E_{12} + c_3 E_{21} + c_4 E_{22} = 0 \Longrightarrow \text{See if } c_1 = c_2 = c_3 = c_4 = 0$$

$$\hookrightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = \cdots = c_4 = 0 \Rightarrow \text{lin. indep.}$$

(ii) does S span V ?

Consider an arbitrary vector (2x2 matrix) in V: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$

See if we can write A as a linear combination of $E_{11}, \dots, E_{22}$:

$$\alpha_1 E_{11} + \alpha_2 E_{12} + \alpha_3 E_{21} + \alpha_4 \alpha_{22} = A$$

$$\hookrightarrow \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \begin{cases} \alpha_1 = a_{11} \\ \vdots \\ \alpha_4 = a_{22} \end{cases} \Rightarrow \text{Yes } S \text{ spans } V.$$

Thus, $E_{11}, \dots, E_{22}$ form a basis for $\mathbb{R}^{2\times 2}$.

Thm If $S = \{v_1, \dots, v_n\}$ spans V, then any collection of $m > n$ vectors in V is linearly dependent.

Corollary If both $S_1 = \{v_1, \dots, v_n\}$ and $S_2 = \{u_1, \dots, u_m\}$ are bases for V, then $n = m$.

EX. Both $S_1 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \right\}$ are bases for $\mathbb{R}^3$.

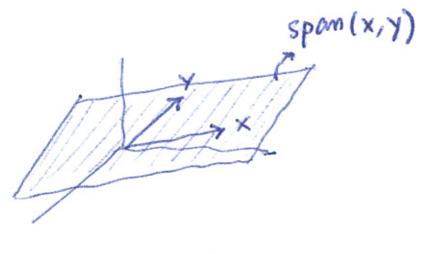
Any basis for $\mathbb{R}^3$ must have exactly 3 elements.

Def. Let V be a vector space.
If V has a basis consisting of n vectors, we say V has dimension n.
V is said to be finite dimensional if there is a finite set of vectors that spans V; otherwise, we say that V is infinite dimensional.
The subspace $\{0\}$ of V is said to have dimension zero.

Consider $V = \mathbb{R}^3$.

① Let x, y, z be 3 linearly indep. vectors in $\mathbb{R}^3$.
Then $\text{span}(x, y, z) = \mathbb{R}^3$ and $\{x, y, z\}$ form a basis for $\mathbb{R}^3$.

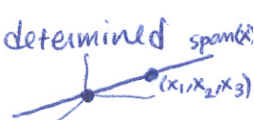
② Let x, y be 2 linearly indep. vectors in $\mathbb{R}^3$.



Then $\underbrace{\text{span}(x, y) = \{\alpha x + \beta y \mid \alpha, \beta \in \mathbb{R}\}}_{\text{a two-dimensional subspace of } \mathbb{R}^3} \subset \mathbb{R}^3$

A vector $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ will be in $\text{span}(x,y)$ iff the point (a,b,c) lies on the plane determined by $(0,0,0)$, (x_1, x_2, x_3), and (y_1, y_2, y_3).
Thus, we can think of a 2D subspace of $\mathbb{R}^3$ as a plane through the origin.

③ Let x be a nonzero vector in $\mathbb{R}^3$.
Then $\text{span}(x) = \{\alpha x \mid \alpha \in \mathbb{R}\} \subset \mathbb{R}^3$ a 1D subspace of $\mathbb{R}^3$
A vector $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ will be in $\text{span}(x)$ iff the point (a,b,c) is on the line determined by $(0,0,0)$ and (x_1, x_2, x_3).
We can represent a 1D subspace of $\mathbb{R}^3$ by a line through the origin



An example of an infinite dimensional vector space:

Let V = the vector space of all polynomials. V is infinite dimensional.

Assume V has dimension n. $\Rightarrow$ V has a basis of n vectors

$\Rightarrow$ any set of $n+1$ vectors would be linearly dep.

However $1, x, x^2, \cdots, x^n$ are linearly indep., since $W[1, x, x^2, \cdots, x^n] > 0$.

$\Rightarrow$ contradiction! $\Rightarrow$ The assumption was wrong.

Similarly, we can show that $C[a,b]$ is infinite dimensional.

__Thm__ If V is a vector space of dimension $n>0$, then

(i) any set of n linearly indep. vectors spans V.

(ii) any n vectors that span V are linearly indep.

__Ex.__ Show that $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

__Way I.__ we show that $\begin{cases} \text{(i) the three vectors are linearly indep.} \Leftrightarrow \det\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = 2 \neq 0 \\ \text{(ii) the three vectors span } \mathbb{R}^3 \end{cases}$

__Way II.__ Since $\dim(\mathbb{R}^3) = \underline{\underline{3}}$, we only need to show that the __three__ vectors are linearly indep., because by the above theorem they will also span $\mathbb{R}^3$ if they are lin. indep.

__Thm__ If V is a vector space of dimension $n>0$, then

(i) no set of fewer than n vectors can span V

(ii) any subset of fewer than n lin. indep. vectors can be extended to form a basis for V.

(iii) any spanning set containing more than n vectors can be pared down to form a basis for V.

__Standard Bases__

We refer to the set $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ as the standard basis for $\mathbb{R}^3$.
(because it is the most simplest & most common one for representing vectors in $\mathbb{R}^3$)
The standard basis for $\mathbb{R}^{2\times 2}$ is $\{E_{11}, E_{12}, E_{21}, E_{22}\}$.
The standard basis for P_n is $\{1, x, x^2, \cdots, x^{n-1}\}$.
__NOTE:__ Standard bases are not necessarily the most appropriate bases. The choice of the basis depends on the particular application.

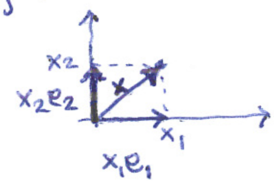
3.5 change of basis

The standard basis for $\mathbb{R}^2$ is $S=\{e_1, e_2\}$ where $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $e_2=\begin{pmatrix}0\\1\end{pmatrix}$.

Any vector $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2$ can be written as a linear comb. $x=\alpha e_1+\beta e_2$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\alpha\begin{pmatrix}1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\alpha\\\beta\end{pmatrix} \implies x=x_1e_1+x_2e_2$$

The scalars x_1 and x_2 can be thought of as the standard coordinates of x, or the coordinates of x w.r.t. the standard basis $\{e_1, e_2\}$.



Sometimes, it will be easier to change the basis or the coordinate system

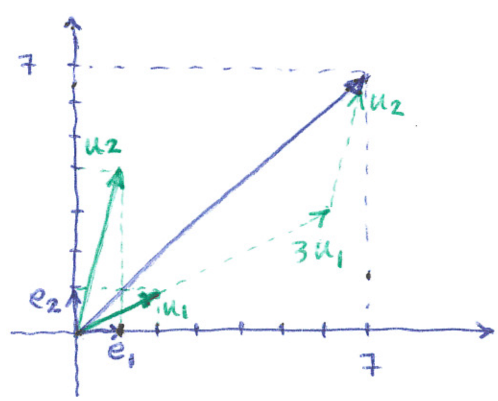
So, instead of $\{e_1, e_2\}$ we use another basis, say $\{u_1, u_2\}$.

Then: $x=\alpha u_1+\beta u_2 \implies \alpha, \beta$ can be uniquely determined

where the scalars α and β are the coordinates of x w.r.t. the basis $\{u_1, u_2\}$

(we refer to $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ as the coordinate vector of x w.r.t. $\{u_1, u_2\}$)

ex $\underbrace{u_1=\begin{pmatrix}2\\1\end{pmatrix}, \quad u_2=\begin{pmatrix}1\\4\end{pmatrix}}_{\text{they form a basis for } \mathbb{R}^2 \text{, because they are lin. indep.}}$, $x=\begin{pmatrix}7\\7\end{pmatrix}$ $\xrightarrow{\text{we have}}$ $x=3u_1+u_2$



Application: Population migration

suppose 30% of the population of a metropolitan area lives in the city & 70% in the suburbs.

each year 6% of the people living in a city move to the suburbs and 2% move from suburbs to the city.

what will be the percentage of population in city & suburbs in n years.

now: $x_0 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \xrightarrow[x_1 = A x_0]{A = \begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}}$ next year: $x_1 = \begin{pmatrix} 0.94 \times 0.3 + 0.02 \times 0.7 \\ 0.06 \times 0.3 + 0.98 \times 0.7 \end{pmatrix}$

After 2 years: $x_2 = A x_1 = A^2 x_0$

After n years: $x_n = A^n x_0$

After many years $\lim_{n\to\infty} x_n = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ called a steady-state vector.

To understand why the sequence (or the process) x_n converges to a steady state, it is helpful to change the coordinate system.

We choose the basis $S = \{u_1, u_2\}$ where $u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for $\mathbb{R}^2$.

These two basis vectors have a nice property: $\begin{cases} A u_1 = u_1 \\ A u_2 = 0.92\, u_2 \end{cases}$ $(*)$

x_0 can be written as a linear comb. of u_1 & u_2:

$$x_0 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 0.05 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.25\, u_1 - 0.05\, u_2$$

$$\Rightarrow \quad x_n = A^n x_0 = A^n (0.25\, u_1 - 0.05\, u_2) = 0.25\, A^n u_1 - 0.05\, A^n u_2$$

$$\{\text{by } (*)\} = 0.25\, u_1 - 0.05\, (0.92)^n u_2$$

$$\lim_{n\to\infty} x_n = 0.25\, u_1 = 0.25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$$

changing coordinates

once we decide to work with a new basis, we need to find the coordinates w.r.t. that basis.

suppose instead of using the standard basis $\{e_1, e_2\}$, we wish to use the basis $\{u_1, u_2\}$ where $u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Sometimes we may need to use both bases and switch back & forth between them.

we have two problems:

(1) Given a vector $c_1 u_1 + c_2 u_2$, find its coordinates w.r.t. e_1 and e_2.

$\downarrow$

we want to switch from $\{u_1, u_2\}$ to $\{e_1, e_2\}$

we need to express u_1 and u_2 in terms of e_1 and e_2: $\begin{cases} u_1 = 3e_1 + 2e_2 \\ u_2 = e_1 + e_2 \end{cases}$

$$\Rightarrow \quad c_1 u_1 + c_2 u_2 = (3c_1 e_1 + 2c_1 e_2) + (c_2 e_1 + c_2 e_2)$$
$$= (3c_1 + c_2)\, e_1 + (2c_1 + c_2)\, e_2$$

The coordinate vector of $c_1 u_1 + c_2 u_2$ w.r.t. $\{e_1, e_2\}$ is $x = \begin{pmatrix} 3c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}}_{U = [u_1 \ u_2]} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

<u>Conclusion</u> Given a coordinate vector $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ w.r.t. $\{u_1, u_2\}$

$x = Uc$ where $U = [u_1 \ u_2]$

↳ transition matrix from $\{u_1, u_2\}$ to $\{e_1, e_2\}$

(2) Given a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, find its coordinates w.r.t. e_1 and e_2.

$c = U^{-1} x$ $\qquad$ U^{-1}: transition matrix from $\{e_1, e_2\}$ to $\{u_1, u_2\}$

↳ exists because its columns are lin. indep.

<u>Ex.</u> $x = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$. $u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the coordinates of x w.r.t. u_1, u_2.

$U = [u_1 \ u_2] = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ $\qquad$ $U^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ $\qquad$ $c = U^{-1} x = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$$\boxed{x = 3u_1 - 2u_2}$$

General case: change from $\{w_1, w_2\}$ of $\mathbb{R}^2$ to $\{u_1, u_2\}$:

Given: $x = c_1 w_1 + c_2 w_2$ (c_1, c_2 given) $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

find: $x = d_1 u_1 + d_2 u_2$ (find d_1, d_2) $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

we have $c_1 w_1 + c_2 w_2 = d_1 u_1 + d_2 u_2 \xrightarrow[U=[u_1\ u_2]]{V=[w_1\ w_2]} Wc = Ud \Rightarrow \boxed{d = U^{-1}Wc}$

transition matrix: $T = U^{-1} W$.

EX. $w_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $w_2 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$T = U^{-1}W = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -4 & -5 \end{pmatrix}$$

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change of basis for a general vector space

Def. Let V be an n-dimensional vector space.
Let $S = \{w_1, \cdots, w_n\}$ be a basis for V.
Any element $v \in V$ can be written as $v = c_1 w_1 + \cdots + c_n w_n$.
the unique vector $c = (c_1, \cdots, c_n)^T \in \mathbb{R}^n$ is called the coordinate vector of v w.r.t. the basis S, denoted by $[v]_S$.
the c_i's are called the coordinates of v relative to S.

EX. $V = \mathbb{R}^3$, $S_1 = \left\{ w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, w_3 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \right\}$ and $S_2 = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

1) Find the transition matrix T from S_1 to S_2
2) Use T to find the coordinates of $x = 3w_1 + 2w_2 - w_3$ w.r.t. S_2.

1) $$T = U^{-1}W = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$$

2) $$[x]_{S_2} = T[x]_{S_1} = \begin{pmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ 3 \end{pmatrix} \Rightarrow x = 8u_1 - 5u_2 + 3u_3$$

NOTE: $3w_1 + 2w_2 - w_3 = 8u_1 - 5u_2 + 3u_3$

<u>In general:</u> Consider an n-dimensional vector space V.

Let $S_1 = \{w_1, w_2, \cdots, w_n\}$ and $S_2 = \{u_1, u_2, \cdots, u_n\}$ be two bases for V,

$$W = [w_1, w_2, \cdots, w_n] \in \mathbb{R}^{n \times n} \qquad U = [u_1, u_2, \cdots, u_n] \in \mathbb{R}^{n \times n}$$

Given any vector $v \in V$, we can write v in terms of both W and V:

$$v = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n$$

$$v = d_1 u_1 + d_2 u_2 + \cdots + d_n u_n$$

Hence, we have:

$$c_1 w_1 + c_2 w_2 + \cdots + c_n w_n = d_1 u_1 + d_2 u_2 + \cdots + d_n u_n$$

$$Wc = Ud$$

where $c = (c_1, \cdots, c_n)^T = [v]_{S_1}$ and $d = (d_1, \cdots, d_n)^T = [v]_{S_2}$

$$W[v]_{S_1} = U[v]_{S_2}$$

Now, if we wish to find $[v]_{S_2}$ we write $[v]_{S_2} = U^{-1} W [v]_{S_1}$

if we wish to find $[v]_{S_1}$ we write $[v]_{S_1} = (U^{-1}W)^{-1} [v]_{S_2}$

Note that both $T = U^{-1}W$ and $T^{-1} = (U^{-1}W)^{-1} = W^{-1}U$ are transition matrices.

Ex. Let $V = P_3$. Consider $S_1 = \{1, x, x^2\}$ and $S_2 = \{1, 2x, 4x^2-2\}$.

Given any $p(x) = c_1 + c_2 x + c_3 x^2$ in P_3, find the coordinates of $p(x)$ w.r.t. S_2.

we need to find $[p(x)]_{S_2} = d = (d_1, d_2, d_3)$ s.t. $p(x) = d_1 \cdot (1) + d_2 \cdot (2x) + d_3 \cdot (4x^2-2)$

we have $$c_1 + c_2 x + c_3 x^2 = d_1 \cdot (1) + d_2 \cdot (2x) + d_3 \cdot (4x^2 - 2)$$

$$\Downarrow$$

$$\begin{aligned} c_1 &= d_1 - 2d_3 \\ c_2 &= 2d_2 \\ c_3 &= 4d_3 \end{aligned}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{T} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$c = Td \quad \Rightarrow \quad d = T^{-1} c = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 + \frac{c_3}{2} \\ c_2/2 \\ c_3/4 \end{pmatrix}$$

$$\Rightarrow p(x) = \left(c_1 + \frac{c_3}{2}\right) \cdot (1) + \left(\frac{c_2}{2}\right) \cdot (2x) + \left(\frac{c_3}{4}\right) \cdot (4x^2 - 2)$$

3.6 Row space & column space

Let $A \in \mathbb{R}^{m \times n}$.

1) the space spanned by the row vectors of A is called the row space of A. The row space is a subspace of $\mathbb{R}^{1 \times n}$.

2) the space spanned by the column vectors of A is called the column space of A. The column space is a subspace of $\mathbb{R}^{m}$.

Ex. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

1) row space of A: $\mathcal{R}(A) = \text{span}\big((1,0,0), (0,1,0) \big) \subset \mathbb{R}^{1 \times 3}$

linear combination of the two row vectors:

$$\alpha(1,0,0) + \beta(0,1,0) = (\alpha, \beta, 0)$$

the collection of all vectors $(\alpha, \beta, 0)$ will form a subspace of $\mathbb{R}^{1 \times 3}$.

2) column space of A: $C(A) = \text{span}\big((1,0)^T, (0,1)^T \big) = \mathbb{R}^2$

linear combination of three column vectors:

$$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The collection of all vectors $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ form the space $\mathbb{R}^2$

Note: zero vectors, such as $(0,0)^T$ do not contribute to the linear combination.

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theorem Two row equivalent matrices have the same row space.

recall: if B is row equivalent to A, then B can be obtained by performing row operations on A.
Hence, the row vectors of B are linear combinations of the row vectors of A.

Def. The rank of a matrix, denoted by $\text{rank}(A)$, is the dimension of the row space of A.

To determine $\text{rank}(A)$, we can reduce A to RE form. The nonzero rows of the RE matrix will form a basis for the row space.

EX. $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

they are row equivalent

then $R(A) = \text{span}\big((1,-2,3), (0,1,5)\big) \subset \mathbb{R}^{1\times 3}$

$\Rightarrow \quad \text{rank}(A) = 2$.

NOTE: The RE form of A is a good way to find $R(A)$, because the RE form of A helps us find the rows that really contribute to $R(A)$.

A few theorems on linear systems $A\mathbf{x} = \mathbf{b}$:

> Recall a theorem from Sec. 1.3:
>
> A linear system $A\mathbf{x}=\mathbf{b}$ is consistent iff $\mathbf{b}$ can be written as a linear combination of the column vectors of A.

$\Downarrow$

Theorem A linear system $A\mathbf{x}=\mathbf{b}$ is consistent iff $\mathbf{b}$ is in the column space of A.

Theorem Let $A \in \mathbb{R}^{m\times n}$. The linear system $A\mathbf{x}=\mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$ iff the column vectors of A span $\mathbb{R}^m$.

NOTE 1) Let $A \in \mathbb{R}^{m\times n}$. If the column vectors of A span $\mathbb{R}^m$, then we must have $n \geq m$, since no set of fewer than m vectors could span $\mathbb{R}^m$.

NOTE 2) Let $A \in \mathbb{R}^{m\times n}$. If the column vectors of A are linearly independent, then we must have $n \leq m$, since every set of more than m vectors in $\mathbb{R}^m$ is linearly dep.

From notes 1 & 2, we conclude:

If the column vectors of $A \in \mathbb{R}^{m\times n}$ form a basis for $\mathbb{R}^m$, then $n = m$.

We will also obtain the following corollaries:

<u>Corollary</u> $A \in \mathbb{R}^{n\times n}$ is nonsingular <u>iff</u> the column vectors of A form a basis for $\mathbb{R}^n$.

<u>Corollary</u> Let $A \in \mathbb{R}^{n\times n}$. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$ <u>iff</u> the column vectors of A form a basis for $\mathbb{R}^n$.

<u>How to find column space of A?</u>

<u>Theorem</u> Let $A \in \mathbb{R}^{m\times n}$. The dimension of $R(A)$ is equal to the dimension of $C(A)$.

Strategy to find $C(A)$:

1) Find the RE form of A, and call it U.
2) Determine the columns of U that correspond to the leading 1's.
3) Find the corresponding columns of A and use them to find $C(A)$.

<u>NOTE:</u> U only tells us which columns of A to use to find $C(A)$.

We cannot use the column vectors of U to find $C(A)$, because in general $C(A) \neq C(U)$.

Recall that we always have $R(A) = R(U)$.

EX. $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$. Find $C(A)$.

1) find the RE form of A: $U = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

2) determine the columns corresponding to the leading 1's: $\begin{cases} \text{column } 1 \\ \text{column } 2 \\ \text{column } 5 \end{cases}$

3) find the corresponding columns of A:

$$C(A) = \text{span}\left((1,-1,0,1)^T, (-2,3,1,2)^T, (2,-2,4,5)^T \right)$$

we also obtain that dimension of $C(A)$ is 3.

NOTE: In the above example, the 3rd & 4th columns of A do not contribute to $C(A)$, because they are linearly dependent with the other three columns and can be written as a linear combination of the 1st, 2nd, and 5th columns of A.

EX. Find the dimension of $S = \text{span}\left((1,2,-1,0)^T, (2,5,-3,2)^T, (2,4,-2,0)^T, (3,8,-5,4)^T \right)$.

S is the same as the column space of $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{pmatrix}$.

we therefor find the dimension of S by finding the dimension of $C(A)$:

1) find the RE form of A: $U = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

2) determine the columns corresponding to the leading 1's: $\begin{cases} \text{column } 1 \\ \text{column } 2 \end{cases}$

3) Hence, dimension of $C(A) = \text{span}\left((1,2,-1,0)^T, (2,5,-3,2)^T \right)$ is two.

only these two vectors form a basis for $C(A)$.

Null space of a matrix

Let $A \in \mathbb{R}^{m\times n}$.

Let $N(A)$ denote the set of all solutions to the homogeneous system $A\mathbf{x}=\mathbf{0}$:

$$N(A) = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \}$$

$N(A)$ is a subspace of $\mathbb{R}^n$:

1) $\mathbf{0} \in N(A)$ because $A\mathbf{0} = \mathbf{0}$ ✓ $\Rightarrow N(A)$ is nonempty

2) if $\mathbf{x} \in N(A)$, $\alpha \in \mathbb{R} \Rightarrow \alpha\mathbf{x} \in N(A)$ because $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}$
   if $\mathbf{x}, \mathbf{y} \in N(A) \Rightarrow \mathbf{x}+\mathbf{y} \in N(A)$ because $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}+\mathbf{0} = \mathbf{0}$

$N(A)$ is called the <u>null space</u> of A.

EX Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$. Determine $N(A)$.

We first solve $A\mathbf{x}=\mathbf{0}$:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right) \longrightarrow \underbrace{\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right)}_{\text{RE form}}$$

Free variables: $\begin{cases} x_3 = \alpha \\ x_4 = \beta \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 + 2x_3 - x_4 = 0 \Rightarrow x_2 = -2\alpha + \beta \end{cases} \Rightarrow x_1 = -\alpha + 2\alpha - \beta = \alpha - \beta$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \alpha-\beta \\ -2\alpha+\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow N(A)$ contains all vectors of the form $\alpha\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ where $\alpha, \beta \in \mathbb{R}$.

Def. The dimension of the null space of A is called the <u>nullity</u> of A.

↳ Nullity of A = the number of free variables.

Theorem (Rank–Nullity Theorem) Let $A \in \mathbb{R}^{m\times n}$. Then rank of A plus nullity of A equals n.

proof. $\text{rank}(A) = \dim(R(A)) = \dim(R(U)) = r$ (no. of nonzero rows of U)

$\Rightarrow \text{rank}(A) = r$ = no. of lead variables

$\Rightarrow n - r$ = no. of free variables = nullity of A

$\Rightarrow n = r +$ nullity of A $\Rightarrow$ $\boxed{n = \text{rank of } A + \text{nullity of } A}$

Ex. $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}$.

1) Find a basis for $R(A)$.

2) Find a basis for $N(A)$.

3) verify that rank of A + nullity of A = 4.

---

1) To find $R(A)$, we first find the RE form of A:

$$U = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows of U form a basis for $R(A)$:

$$R(A) = \text{span}\left((1,2,0,3), (0,0,1,2) \right)$$

and A has rank 2.

2) To find $N(A)$, we need to solve $A\mathbf{x} = \mathbf{0}$, which is equivalent to the system $U\mathbf{x} = 0$:

$$\begin{cases} x_1 + 2x_2 \qquad + 3x_4 = 0 \\ \qquad\qquad x_3 + 2x_4 = 0 \end{cases} \quad (*)$$

free variables: $\begin{cases} x_2 = \alpha \\ x_4 = \beta \end{cases} \overset{(*)}{\Longrightarrow} \begin{cases} x_1 = -2\alpha - 3\beta \\ x_3 = -2\beta \end{cases}$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} -2\alpha - 3\beta \\ \alpha \\ -2\beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow N(A) = \text{span}\left((-2,1,0,0)^T, (-3,0,2,1)^T \right)$$

and A has nullity 2.

3) Since rank of A is 2 and nullity of A is 2, then

rank of A + nullity of A = 2 + 2 = 4.