Solutions to HW #7

Math 314, Spring 2017

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① $x = \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$, $y = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$

a) $$\cos\theta = \frac{x^T y}{\|x\| \cdot \|y\|} = \frac{(-3)\times 0 + 2\times 3 + 6\times(-4)}{\sqrt{(-3)^2+2^2+6^2} \times \sqrt{0^2+3^2+(-4)^2}} = \frac{-18}{7\times 5} = \frac{-18}{35}$$

Hence $\theta = \cos^{-1}\left(\frac{-18}{35}\right)$.

b) $|x^T y| = |-18| = 18$

$\|x\| \cdot \|y\| = 7 \times 5 = 35$

$$\xRightarrow{18 \le 35} |x^T y| \le \|x\| \cdot \|y\|$$

c) $$p = \frac{x^T y}{y^T y}\, y = \frac{-18}{0^2+3^2+(-4)^2}\begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \frac{-18}{25}\begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ -54/25 \\ 72/25 \end{pmatrix}$$

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② $X = \operatorname{span}\left(\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right) \subset \mathbb{R}^3$

a) $X^\perp = \left\{ z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z^T x = 0, \quad x \in X \right\}$

A general vector x in the vector space X will have the form:

$$x = \alpha \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\beta \\ 0 \\ 2\alpha \end{pmatrix} \qquad \text{where } \alpha, \beta \in \mathbb{R}.$$

We need to find all $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ such that $z^T x = 0$. This means

$$z^T x = (z_1 \;\; z_2 \;\; z_3) \begin{pmatrix} -\beta \\ 0 \\ 2\alpha \end{pmatrix} = -\beta z_1 + 2\alpha z_3 = 0, \qquad \forall \alpha, \beta \in \mathbb{R}.$$

The only choice of z_1 and z_3 such that $-\beta z_1 + 2\alpha z_2 = 0$ for any scalars α and β is that $z_1 = z_3 = 0$. this means that vector z must have the form : $z = \begin{pmatrix} 0 \\ z_2 \\ 0 \end{pmatrix}$.

Hence, $X^{\perp} = \left\{ \begin{pmatrix} 0 \\ z_2 \\ 0 \end{pmatrix}, \ z_2 \in \mathbb{R} \right\} = \text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = \text{span}(e_2)$.

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b) $X = \text{span}\left(\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right) = \text{span}(e_3, e_1)$

$X^{\perp} = \text{span}(e_2)$

$X \oplus X^{\perp} = \text{span}(e_3, e_1, e_2) = \mathbb{R}^3$

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x_i	0	$\pi/6$	$\pi/4$	$\pi/3$
y_i	2	1	1	0.5

a) $y = \alpha + \beta x \implies \begin{cases} \alpha + 0 \times \beta = 2 \\ \alpha + \pi/6 \times \beta = 1 \\ \alpha + \pi/4 \times \beta = 1 \\ \alpha + \pi/3 \times \beta = 0.5 \end{cases} \implies \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & \pi/6 \\ 1 & \pi/4 \\ 1 & \pi/3 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{a} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0.5 \end{bmatrix}}_{b}$

$Aa = b \implies$ The least squares solution $a = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is given by

$$A^T A a = A^T b \implies \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \pi/6 & \pi/4 & \pi/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & \pi/6 \\ 1 & \pi/4 \\ 1 & \pi/3 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \pi/6 & \pi/4 & \pi/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0.5 \end{bmatrix}$$

$$\implies \begin{bmatrix} 4 & 3\pi/4 \\ 3\pi/4 & \frac{29\pi^2}{144} \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 4.5 \\ 7\pi/12 \end{pmatrix} \implies \begin{cases} \alpha \approx 1.9286 \\ \beta \approx -1.3642 \end{cases}$$

$$\implies \boxed{y = 1.9286 - 1.3642\, x}$$

b) $y = \alpha + \beta \sin x \Rightarrow \begin{cases} \alpha + \sin(0) \times \beta = 2 \\ \alpha + \sin(\pi/6) \times \beta = 1 \\ \alpha + \sin(\pi/4) \times \beta = 1 \\ \alpha + \sin(\pi/3) \times \beta = 0.5 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1/2 \\ 1 & \sqrt{2}/2 \\ 1 & \sqrt{3}/2 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{a} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0.5 \end{bmatrix}}_{b}$

$Aa = b \Rightarrow$ The least squares solution $a = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is given by

$$A^T A a = A^T b \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/2 & \sqrt{2}/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1/2 \\ 1 & \sqrt{2}/2 \\ 1 & \sqrt{3}/2 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/2 & \sqrt{2}/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0.5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 2.0731 \\ 2.0731 & 1.5 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 4.5 \\ 1.6401 \end{pmatrix} \Rightarrow \begin{cases} \alpha \approx 1.9680 \\ \beta \approx -1.6266 \end{cases}$$

$$\Rightarrow \boxed{y = 1.9680 - 1.6266 \sin x}$$

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4. $x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad y = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

We first notice that $x^T y = 2 \times 3 + (-1) \times 6 = 0 \Rightarrow x \perp y$

a) Pythagorean law will not hold in infinity norm, because:

$\|x\|_\infty^2 + \|y\|_\infty^2 = (\max(2,1))^2 + (\max(3,6))^2 = 2^2 + 6^2 = 40$ $\xRightarrow{40 \neq 25} \|x+y\|_\infty^2 \neq \|x\|_\infty^2 + \|y\|_\infty^2$

$\|x+y\|_\infty^2 = (\max(|2+3|, |-1+6|))^2 = (\max(5,5))^2 = 5^2 = 25$

b) Pythagorean law will hold in 2-norm, because:

$\|x\|_2^2 + \|y\|_2^2 = (2^2 + (-1)^2) + (3^2 + 6^2) = 5 + 45 = 50$ $\Rightarrow \|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$

$\|x+y\|_2^2 = (2+3)^2 + (-1+6)^2 = 5^2 + 5^2 = 25 + 25 = 50$