Solutions to HW #6

Math 314, Spring 2017
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① a) $L: \mathbb{R}^2 \to \mathbb{R}^2$, $\quad L(x) = \begin{pmatrix} x_1 + x_2 \\ 3x_2 \end{pmatrix}$, $\quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

L is a linear transformation, because $\forall \alpha, \beta \in \mathbb{R}$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $y = \begin{pmatrix} y_1 \\ y \end{pmatrix} \in \mathbb{R}^2$ we have:

$$L(\alpha x + \beta y) = \begin{pmatrix} \alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2 \\ 3(\alpha x_2 + \beta y_2) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(x_1 + x_2) + \beta(y_1 + y_2) \\ \alpha(3x_2) + \beta(3y_2) \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_1 + x_2 \\ 3x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 + y_2 \\ 3y_2 \end{pmatrix} = \alpha L(x) + \beta L(y)$$

b) $L: \mathbb{R} \to \mathbb{R}$, $\quad L(x) = |x|$, $\quad x \in \mathbb{R}$

L is not linear because for $x \in \mathbb{R}$ and $y \in \mathbb{R}$:

$$L(x+y) = |x+y| \neq |x| + |y| = L(x) + L(y)$$

c) $L: P_2 \to \mathbb{R}$, $\quad L(p) = \int_a^b p(x)\, dx$, $\quad p \in P_2$

L is a linear transformation, because $\forall \alpha, \beta \in \mathbb{R}$ and $p \in P_2$ and $q \in P_2$:

$$L(\alpha p + \beta q) = \int_a^b [\alpha p(x) + \beta q(x)]\, dx$$

$$= \int_a^b \alpha p(x)\, dx + \int_a^b \beta q(x)\, dx$$

$$= \alpha \int_a^b p(x)\, dx + \beta \int_a^b q(x)\, dx = \alpha L(p) + \beta L(q).$$

② $L: \mathbb{R}^2 \to \mathbb{R}$, $L(x) = x_1 + x_2$, $\forall\, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

a) Ker(L)

$$\mathrm{Ker}(L) = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; L(x) = 0 \right\}$$

$$= \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 + x_2 = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix},\ x_1 \in \mathbb{R} \right\}$$

$$= \left(\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \alpha \in \mathbb{R} \right)$$

$$= \mathrm{span}\left((1,-1)^T \right)$$

b) range (L)

$$\mathrm{range}(L) = \left\{ y \in \mathbb{R} \;\middle|\; y = L(x),\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \right\}$$

$$= \left\{ y = x_1 + x_2,\ x_1, x_2 \in \mathbb{R} \right\}$$

$$= \left\{ y = \alpha,\ \alpha \in \mathbb{R} \right\}$$

$$= \mathbb{R}$$

c) Since $\mathrm{rang}(L) = \mathbb{R}$, the transformation is <u>onto</u>.

(3) $E = \{e_1, e_2, e_3\}$ basis for $\mathbb{R}^3$

$F = \left\{ b_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, b_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ basis for $\mathbb{R}^2$

$L : \mathbb{R}^3 \to \mathbb{R}^2$, $\quad L(x) = (x_1 - x_2)\, b_1 + (x_1 - x_3)\, b_2$, $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$

We need to find $A \in \mathbb{R}^{2\times 3}$ s.t. $[L(x)]_F = A\,[x]_E$.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$.

$a_1 = [L(e_1)]_F$ where $L(e_1) = (1-0)\, b_1 + (1-0)\, b_2 = b_1 + b_2$

$\Rightarrow \quad a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$a_2 = [L(e_2)]_F$ where $L(e_2) = (0-1)\, b_1 + (0-0)\, b_2 = -b_1 + 0 \times b_2$

$\Rightarrow \quad a_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$a_3 = [L(e_3)]_F$ where $L(e_3) = (0-0)\, b_1 + (0-1)\, b_2 = 0 \times b_1 - b_2$

$\Rightarrow \quad a_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Hence, $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

④ $L: \mathbb{R}^3 \to \mathbb{R}^3$, $\quad L(x) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_1 \end{pmatrix}$, $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.

a) Find $A \in \mathbb{R}^{3\times 3}$ s.t. $[L(x)]_E = A\,[x]_E$ where $E = \{e_1, e_2, e_3\}$.

Let $A = [a_1 \quad a_2 \quad a_3]$

$a_1 = [L(e_1)]_E$ where $L(e_1) = \begin{pmatrix} 1+0 \\ 0+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 \times e_1 + 0 \times e_2 + 1 \times e_3$

$\Rightarrow a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$a_2 = [L(e_2)]_E$ where $L(e_2) = \begin{pmatrix} 0+1 \\ 1+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \times e_1 + 1 \times e_2 + 0 \times e_3$.

$\Rightarrow a_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$a_3 = [L(e_3)]_E$ where $L(e_3) = \begin{pmatrix} 0+0 \\ 0+1 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \times e_1 + 1 \times e_2 + 1 \times e_3$

$\Rightarrow a_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Hence, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

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b) Find $B \in \mathbb{R}^{3\times 3}$ s.t. $[L(x)]_F = B\,[x]_F$ where $F = \left\{u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.

Let $B = [b_1 \quad b_2 \quad b_3]$

$b_1 = [L(u_1)]_F$ where $L(u_1) = \begin{pmatrix} 1+2 \\ 2+3 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$

to find $\left[\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}\right]_F$, we need to multiply $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ by $U^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 1/2 & -1/3 \\ 1 & -1/2 & 0 \end{bmatrix}$

$\Rightarrow b_1 = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1/2 & -1/3 \\ 1 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 7/6 \\ 1/2 \end{pmatrix}$

$$b_2 = [L(u_2)]_F \quad \text{where} \quad L(u_2) = \begin{pmatrix} 1+2 \\ 2+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow b_2 = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1/2 & -1/3 \\ 1 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2 \end{pmatrix}$$

$$b_3 = [L(u_3)]_F \quad \text{where} \quad L(u_3) = \begin{pmatrix} 1+0 \\ 0+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow b_3 = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1/2 & -1/3 \\ 1 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ 1 \end{pmatrix}$$

Hence, $$B = \begin{bmatrix} 4/3 & 1/3 & 1/3 \\ 7/6 & 2/3 & -1/3 \\ 1/2 & 2 & 1 \end{bmatrix}$$

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c) S: transition matrix to change basis from F to E:

given $x = \alpha u_1 + \beta u_2 + \gamma u_3$, find $x = a e_1 + b e_2 + c e_3$

This means $$\underbrace{[u_1 \; u_2 \; u_3]}_{S} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \underbrace{[e_1 \; e_2 \; e_3]}_{I} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = S \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{where} \quad S = [u_1 \; u_2 \; u_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

We finally verify that $SB = AS$:

$$SB = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4/3 & 1/3 & 1/3 \\ 7/6 & 2/3 & -1/3 \\ 1/2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

$$AS = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

$\Rightarrow SB = AS \Rightarrow B = S^{-1}AS$.