Sec. 3.3, 3.4, 3.5, 3.6

① Determine whether the following vectors are linearly independent:

a) $x = (1, 3, 0)^T$, $y = (-1, 1, 1)^T$, $z = (0, 2, 4)^T$

b) $x = (1, -1, 2)^T$, $y = (1, 0, 2)^T$

c) $x = (1, 0, 2, 1)^T$, $y = (-2, 2, 0, -1)^T$, $z = (1, 1, 4, -3)^T$

d) $p_1(x) = x^2 - 2x + 1$, $p_2(x) = 2x^2 - 3$, $p_3(x) = -x^2 - x$

e) $f_1(x) = x^2$, $f_2(x) = x|x|$

f) $f_1(x) = x^3$, $f_2(x) = x^2|x|$

g) $f_1(x) = e^{2x}$, $f_2(x) = \sin(x)$, $f_3(x) = x^2 + 1$

② If $S = \{v_1, v_2, v_3, v_4\}$ <u>spans</u> a vector space V, then what we can say about the linear dependency between

a) v_1, v_2, v_3,
b) v_1, v_2, v_3, v_4
b) v_1, v_2, v_3, v_4, v_5 where $v_5 \in V$

③ If $S = \{v_1, v_2, v_3, v_4\}$ is a <u>basis</u> for a vector space V, then what we can say about the linear dependency between

a) v_1, v_2, v_3
b) v_1, v_2, v_3, v_4
c) v_1, v_2, v_3, v_4, v_5 where $v_5 \in V$

(4) Determine the dimension of the following vector spaces:

a) $V = \mathbb{R}^4$

b) $V =$ vector space of all polynomials

c) $V =$ vector space of all polynomials of degree at most 4.

d) $V =$ vector space of all continuous functions

e) $V = \text{span}\left((1,1,2)^T, (0,1,2)^T, (2,2,4)^T \right)$

(5) What do we mean by saying a vector space V has dimension n?

(6) Consider a vector $x = (2,4)^T$. Find the coordinates of x with respect to $u_1 = (2,3)^T$ and $u_2 = (0,1)^T$.

(7) Find the transition matrix T from $E = \{u_1, u_2\}$ to $F = \{v_1, v_2\}$ where $u_1 = (1,1)^T$, $u_2 = (-2,0)^T$, $v_1 = (0,2)^T$, $v_2 = (-1,1)^T$.

(8) In question 7, what is the transition matrix from F to E?

(9) Consider the following two bases for $V = \mathbb{R}^3$:

$E = \{v_1, v_2, v_3\}$, where $v_1 = (1,1,1)^T$, $v_2 = (1,2,3)^T$, $v_3 = (0,-1,1)^T$

$F = \{u_1, u_2, u_3\}$, where $u_1 = (1,1,0)^T$, $u_2 = (-1,1,1)^T$, $u_3 = (0,1,-1)^T$

Let now $x \in \mathbb{R}^3$ be a vector in $\mathbb{R}^3$, and let $[x]_E$ and $[x]_F$ denote the coordinate vectors of x with respect to the bases E and F, respectively.

a) Find the matrix T such that $[x]_F = T\,[x]_E$

b) If $x = v_1 + v_2 - v_3$, find (α, β, γ) such that $x = \alpha u_1 + \beta u_2 + \gamma u_3$.

(10) Consider the following two bases for $V = P_3$:

$E = \{1, x, 2x^2\}$ and $F = \{2x, x-1, x^2-2x\}$.

Let $p(x) = -1 + 2x^2$.

a) Find the coordinates of $p(x)$ with respect to E.

b) Find the coordinates of $p(x)$ with respect to F.

c) Write a relation in matrix form between the coordinate vectors in parts a and b.

(11) Find the rank of $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(12) Consider $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -2 \end{pmatrix}$.

a) Find the row space $R(A)$.

b) Find the column space $C(A)$.

c) Determine the rank of A.

(13) Determine the dimension of the space $S = \text{span}\left((1,2,-1)^T, (1,-3,2)^T, (2,-1,1)^T\right)$.

(14) Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

a) Find the null space $N(A)$

b) Determine the nullity of A.

c) Determine the rank of A.

(15) Select and mark the correct check-box. No discussion/explanation is needed.

a) A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

☐ True ☐ False

b) Let $A \in \mathbb{R}^{m\times n}$. For the linear system $A\mathbf{x} = \mathbf{b}$ to be consistent for every vector $\mathbf{b} \in \mathbb{R}^m$, the rows of A must span $\mathbb{R}^n$.

☐ True ☐ False

c) The matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular if and only if the column vectors of A form a basis for $\mathbb{R}^n$.

☐ True ☐ False

d) Let $A \in \mathbb{R}^{3\times 2}$. It is impossible for the column vectors of A to span $\mathbb{R}^3$.

☐ True ☐ False

e) Let $A \in \mathbb{R}^{3\times 2}$. It is impossible for the column vectors of A to be linearly independent.

☐ True ☐ False

f) Consider $A \in \mathbb{R}^{m\times n}$ and let U be the RE form of A.

1) $R(A) = R(U)$. ☐ True ☐ False

2) $C(A) = C(U)$. ☐ True ☐ False

3) rank of A = dimension of $R(U)$. ☐ True ☐ False

4) rank of A + nullity of A $= m$. ☐ True ☐ False

Sec. 4.1, 4.2, 4.3

① Consider the mapping $L(\mathbf{x}) = x_1 + x_2$, $\forall\, \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$.

Show that $L: \mathbb{R}^2 \to \mathbb{R}$ is a linear transformation.

② Consider the mapping $L(\mathbf{x}) = (x_1 + x_2, x_2)$, $\forall\, \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$.

Show that $L: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator.

③ Determine whether the following mappings are linear or nonlinear transformations:

a) $L(\mathbf{x}) = \sin(x_1) + \sin(x_2)$, $\forall\, \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$

b) $L(\mathbf{x}) = A^2 \mathbf{x}$, $A \in \mathbb{R}^{3\times 3}$, $\forall\, \mathbf{x} \in \mathbb{R}^3$

c) $L(f) = \int_a^b |f(x)|\, dx$, $\forall\, f \in C[a,b]$

④ Consider the linear operator $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$L(\mathbf{x}) = (x_1 + x_2, 0)^T, \quad \forall\, \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2.$$

a) Find $\ker(L)$.

b) Find range(L).

c) Is the operator L onto ?

⑤ Consider the linear transformation $D: P_3 \to P_2$ defined by

$$D(p(x)) = p'(x), \quad \forall\, p \in P_3.$$

a) Find $\ker(D)$.

b) Find range(D).

c) Is the transformation D onto ?

(6) Consider the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L(x) = Ax \quad , \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \quad , \quad \forall\, x \in \mathbb{R}^2 .$$

a) Find $\ker(L)$.

b) Find range(L).

c) Is the transformation L onto?

(7) Consider the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$L(x) = (x_1 \,,\, x_2 - x_3)^T \quad , \quad \forall\, x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 .$$

Find the matrix representing L, that is, find the matrix A such that $L(x) = Ax$, $\forall\, x \in \mathbb{R}^3$.

(8) Consider the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$L(x) = (x_1 - x_2 - x_3 \,,\, x_1 + x_2)^T \quad , \quad \forall\, x \in (x_1, x_2, x_3)^T \in \mathbb{R}^3 .$$

Find the matrix A representing L with respect to the bases $E = \{e_1, e_2, e_3\}$ and $F = \{u_1, u_2\}$ where $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

<u>Hint:</u> You need to find A such that $[L(x)]_F = A\,[x]_E$, $\forall\, x \in \mathbb{R}^3$.

(9) Consider the linear transformation $D : P_3 \to P_2$ defined by

$$D(p(x)) = p'(x) \quad , \quad \forall\, p \in P_3 .$$

Find the matrix A representing D with respect to the bases $E = \{2, x+1, -x^2\}$ and $F = \{-2x + 2 + 4x\}$.

<u>Hint:</u> You need to find A such that $[D(p(x))]_F = A\,[p(x)]_E$, $\forall\, p \in P_3$

(10) Consider the linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$L(x) = \begin{pmatrix} x_1 - x_2 \\ 2x_2 \end{pmatrix} \quad \text{for} \quad \forall\, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

a) Find the matrix A representing L with respect to the basis $E = \{e_1, e_2\}$.

b) Find the matrix B representing L with respect to the basis $F = \{u_1, u_2\}$ where $u_1 = (1, 1)^T$ and $u_2 = (-2, 2)^T$.

c) Find the matrix C representing L with respect to the basis $G = \{v_1, v_2\}$ where $v_1 = (0, 4)^T$ and $v_2 = (3, 1)^T$.

d) Find a relation in matrix form between A and B.

e) Find a relation in matrix form between A and C.

f) Find a relation in matrix form between B and C.

g) Are all three matrices A, B, C similar?

(11) Consider the linear operator $D: P_3 \to P_3$ defined by

$$D(p(x)) = 2\,p(x) - p'(x), \qquad \forall\, p \in P_3.$$

a) Find the matrix A representing D with respect to $E = \{1, x, x^2\}$.

b) Find the matrix B representing D with respect to $F = \{x, 2x^2, 2x - 1\}$.

c) Find the transition matrix S to change basis from F to E.

d) Verify that $B = S^{-1} A S$.

Sec. 5.1, 5.2, 5.3, 5.4

① Consider two vectors $x = (1, 2, 4)^T$ and $y = (0, 1, -1)^T$.

a) Find the scalar product of x and y.

b) Find the Euclidean length of x.

c) Find the Euclidean length of y.

d) Find the distance between x and y.

e) Find the angle θ between x and y.

f) Verify that the Cauchy-Schwarz inequality holds for x and y.

g) Find the vector projection P of x onto y.

② Let $X = \text{span}\left((a, 0, 1)^T, (2, 2, 1)^T \right)$ and $Y = \text{span}\left((1, -1, 0)^T \right)$ be two subspaces of $\mathbb{R}^3$. Find $a \in \mathbb{R}$ such that $X \perp Y$.

③ Determine whether the following two subspaces of $\mathbb{R}^3$ are orthogonal:

$$X = \text{span}\left((1, -1, -1)^T, (0, 0, 1)^T \right) \text{ and } Y = \text{span}\left((1, 1, 0)^T \right).$$

④ Let $X = \text{span}\left((0, 0, 1)^T, (-1, 0, 0)^T, (-1, 0, 1)^T \right)$ be a subspace of $\mathbb{R}^3$.

a) Find the orthogonal complement $X^\perp$ of X.

b) Verify that $X \oplus X^\perp = \mathbb{R}^3$.

⑤ Let $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.

a) Find $N(A)$ and $C(A^T)$.

b) Verify that $N(A) \oplus C(A^T) = \mathbb{R}^3$.

(6) Find the least squares solution to the system

$$\begin{cases} x_1 - x_2 = 1 \\ -x_1 + 2x_2 = 0 \\ 2x_1 + 3x_2 = 2 \\ -x_1 - x_2 = 4 \end{cases}$$

(7) Consider the following data points:

i	1	2	3
x_i	0	2	4
y_i	-1	2	4

a) Find the best least squares fit by a linear function $y = \alpha + \beta x$.

b) Find the 2-norm of the residual vector obtained by the model in a).

c) Find the best least squares fit by a quadratic function $y = \alpha + \beta x^2$.

d) Find the 2-norm of the residual vector obtained by the model in c).

e) Based on your findings, which model is better?

(8) Consider two vectors $x = (2,2,1)^T$ and $y = (-2,0,4)^T$ in $\mathbb{R}^3$.

a) Verify that the phytagorean law holds in 2-norm.

b) verify that the phytagorean law does not hold in 1-norm.

(9) Let $V = C[0,1]$ with the inner product $\langle f,g \rangle = \int_0^1 2 f(x)\, g(x)\, dx$.

a) Determine whether the vectors 1 and $x - \frac{1}{2}$ are orthogonal.

b) Determine whether the pythagorean law holds for 1 and $x - \frac{1}{2}$.

(10) Let $V = \mathbb{R}^2$ with the inner product $\langle x,y \rangle = 0.2\, x_1 y_1 + 0.8\, x_2 y_2$.
Let $\|\cdot\|$ denote the norm defined by the inner product $\langle x,y \rangle$.
verify the Cauchy-Schwarz inequality for $x = (2,1)^T$ and $y = (-3,2)^T$.

Sec. 6.1

① Find the eigen pairs of $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$.

② Find the eigenpairs of $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.