Examples of Ch. 3

EX. 1 Show that $\mathbb{R}^2$ together with usual addition and scalar multiplication of two-dimensional vectors satisfies the eight axioms of a vector space.

A1. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}, \quad \forall \mathbf{x},\mathbf{y} \in V$

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$

then $\mathbf{x}+\mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$

$\mathbf{y}+\mathbf{x} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1+x_1 \\ y_2+x_2 \end{pmatrix}$

$\Longrightarrow \quad \mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$

A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z}), \quad \forall \mathbf{x},\mathbf{y},\mathbf{z} \in V$

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ and $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2$

then $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1+z_1 \\ x_2+y_2+z_2 \end{pmatrix}$

$\mathbf{x}+(\mathbf{y}+\mathbf{z}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1+z_1 \\ y_2+z_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1+z_1 \\ x_2+y_2+z_2 \end{pmatrix}$

$\Longrightarrow (\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$

A3. $\exists$ an element $\mathbf{0} \in V$ s.t. $\mathbf{x}+\mathbf{0} = \mathbf{x}, \quad \forall \mathbf{x} \in V$

Let $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

then $\mathbf{x}+\mathbf{0} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+0 \\ x_2+0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x}$

A4. $\forall \mathbf{x} \in V, \exists -\mathbf{x} \in V$ s.t. $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V$ and $-\mathbf{x} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \in V$.

Then $\mathbf{x}+(-\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_1-x_1 \\ x_2-x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

The other four axioms are left as an exercise for you.

$\boxed{\text{EX. 2}}$ Describe the vector space P_3.

P_3 is the set of all polynomials of degree at most 3.
A general polynomial of degree at most 3 has the form

$$p(x) = a_3 x^2 + a_2 x + a_3 \quad \text{where } a_1, a_2, a_3 \in \mathbb{R}$$

We can write

$$P_3 = \left\{ p(x) = a_3 x^2 + a_2 x + a_3 \;\middle|\; a_1, a_2, a_3 \in \mathbb{R} \right\}$$

The following two operations are defined for P_3:

1) scalar multiplication: $(\alpha p)(x) = \alpha \cdot p(x)$, $\alpha \in \mathbb{R}$, $p \in P_3$
2) addition: $(p+q)(x) = p(x) + q(x)$, $p, q \in P_3$

The zero element of P_3 is given by:

$$z(x) = 0 \cdot x^2 + 0 \cdot x + 0 .$$

$\boxed{\text{EX. 3}}$ Let $V = \mathbb{R}^2$. Determine whether the following set forms a subspace of V:

$$S = \left\{ \mathbf{x} = (x_1, x_2)^T \;\middle|\; x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_2 = 2x_1 \right\} .$$

We need to show: 1) S is a nonempty subset of V.
2) $\alpha \mathbf{x} \in S$ whenever $\mathbf{x} \in S$ and $\forall \alpha \in \mathbb{R}$
3) $\mathbf{x} + \mathbf{y} \in S$ whenever $\mathbf{x} \in S$ and $\mathbf{y} \in S$

1) Obviously S is a subset of $\mathbb{R}^2$ because it contains two-dimensional vectors. Moreover, S is nonempty because $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$ since $0 = 2 \cdot 0 = 0$.

2) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} \in S$. Then $\alpha \mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha \cdot 2x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 2(\alpha x_1) \end{pmatrix} \in S$.

3) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} \in S$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ 2y_1 \end{pmatrix} \in S$. Then $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} y_1 \\ 2y_1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 2(x_1 + y_1) \end{pmatrix} \in S$

Hence, S is a subspace of $V = \mathbb{R}^2$.

EX. 4 Let $V = \mathbb{R}^3$. Determine whether the following set forms a subspace of V:

$$S = \{ \mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = x_2 \}$$

1) Obviously S is a subset of $\mathbb{R}^3$ because it contains three-dimensional vectors. Moreover, S is nonempty because $\mathbf{0} = (0,0,0)^T \in S$ since the first two entries of $\mathbf{0} = (0,0,0)^T$ are equal.

2) let $\mathbf{x} = \begin{pmatrix} a \\ a \\ b \end{pmatrix} \in S$. then $\forall \alpha \in \mathbb{R}$, $\alpha \mathbf{x} = \begin{pmatrix} \alpha a \\ \alpha a \\ \alpha b \end{pmatrix} \in S$

3) let $\mathbf{x} = \begin{pmatrix} a \\ a \\ b \end{pmatrix} \in S$ and $\mathbf{y} = \begin{pmatrix} c \\ c \\ d \end{pmatrix} \in S$. then $\mathbf{x} + \mathbf{y} = \begin{pmatrix} a \\ a \\ b \end{pmatrix} + \begin{pmatrix} c \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ a+c \\ b+d \end{pmatrix} \in S$

Hence, S is a subspace of $V = \mathbb{R}^3$.

EX. 5 Let $V = \mathbb{R}^{2\times 2}$. Show that $S = \{ A = (a_{ij}) \in \mathbb{R}^{2\times 2} \mid a_{21} = 0 \}$ is a subspace of V.

1) S is a subset of V and nonempty because $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$ since its entry in 2nd row and 1st column is zero.

2) $\forall\ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$ and $\forall \alpha \in \mathbb{R}$, $\alpha A = \begin{pmatrix} \alpha a & \alpha b \\ 0 & \alpha c \end{pmatrix} \in S$

3) $\forall\ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$ and $\forall\ B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \in S$, $A + B = \begin{pmatrix} a+d & b+e \\ 0 & c+f \end{pmatrix} \in S$.

EX. 6 Let $V = P_2$. Show that $S = \{ p(x) \in P_2 \mid p(0) = 0 \}$ is a subspace of V.

1) S is a subset of V and nonempty because it contains the zero polynomial

$z(x) = 0 \cdot x + 0$ since $z(0) = 0$.

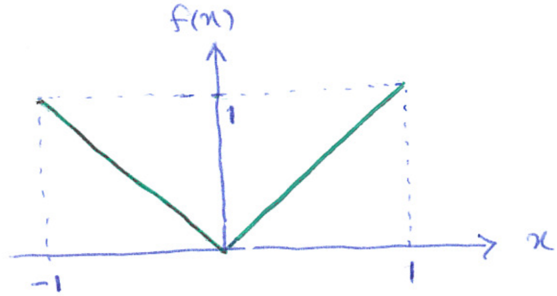
2) $\forall\ p(x) \in S$ and $\alpha \in \mathbb{R}$, $(\alpha p)(0) = \alpha \cdot p(0) = \alpha \cdot 0 = 0 \Rightarrow (\alpha p)(x) \in S$.

3) $\forall\ p(x) \in S$ and $q(x) \in S$, $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0 \Rightarrow (p+q)(x) \in S$.

Hence, S is a subspace of V.

$\boxed{\text{Ex. 7}}$ By giving an example show that $C([a,b])$ is not a subspace of $C^1([a,b])$

For example consider the function $f(x) = |x|$ on $[-1, 1]$



This function is continuous on $[-1,1]$, hence it belongs to $C[-1,1]$.

But it is not continuously differentiable, because $f'(x) = \frac{|x|}{x}$ is not defined at $x=0$. Hence $f'(x)$ is not continuous at $x=0$, and therefore $f'(x)$ does not belong to $C[-1,1]$, which means that $f(x)$ does not belong to $C^1[-1,1]$.

To summarize $f(x) = |x|$ belongs to $C[-1,1]$ but does not belong to $C^1[-1,1]$.

Hence $C[-1,1]$ is not a subset of $C^1[-1,1]$.

Hence $C[-1,1]$ is not a subspace of $C^1[-1,1]$.

$\boxed{\text{Remark}}$ The converse is true; that is, $C^1([a,b])$ is a subspace of $C([a,b])$.

In general $C^n([a,b])$ is a subspace of $C([a,b])$, because every n-times continuously differentiable function is a continuous function.

Also, P_n is a subspace of $C([a,b])$, because every polynomial of degree at most $n-1$ is a continuous function.

$\boxed{\text{EX. 8}}$ Let $\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. What is $\text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$?

$\text{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the set of all linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{x} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \begin{pmatrix}\alpha_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\alpha_3\end{pmatrix} = \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

The set of all vectors $\mathbf{x} = (\alpha_1, \alpha_2, \alpha_3)$ with real entries is the vector space $\mathbb{R}^3$. Hence $\text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

$\boxed{\text{EX. 9}}$ Let $\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$. What is $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$.

$\text{Span}(\mathbf{e}_1, \mathbf{e}_2)$ is the set of all linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{x} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 = \begin{pmatrix}\alpha_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\\0\end{pmatrix} = \begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix} \quad \text{where } \alpha_1, \alpha_2 \in \mathbb{R}.$$

Since the third entry of $\mathbf{x}$ is always zero, hence $\text{span}(\mathbf{e}_1, \mathbf{e}_2) \neq \mathbb{R}^3$.

However, we can show that $S = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$ is a subspace of $\mathbb{R}^3$, because:

1) the elements of S are vectors of form $\begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix}$. Hence S is a subset of $\mathbb{R}^3$.

2) $\mathbf{0} = \begin{pmatrix}0\\0\\0\end{pmatrix} \in S$ because its third entry is zero. Hence S is nonempty.

3) if $\mathbf{x} = \begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix} \in S$ and $\beta \in \mathbb{R}$, then $\beta\mathbf{x} = \begin{pmatrix}\beta\alpha_1\\\beta\alpha_2\\\beta\cdot 0\end{pmatrix} = \begin{pmatrix}\beta\alpha_1\\\beta\alpha_2\\0\end{pmatrix} \in S$

4) if $\mathbf{x} = \begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix} \in S$ and $\mathbf{y} = \begin{pmatrix}\gamma_1\\\gamma_2\\0\end{pmatrix} \in S$, then $\mathbf{x}+\mathbf{y} = \begin{pmatrix}\alpha_1+\gamma_1\\\alpha_2+\gamma_2\\0\end{pmatrix} \in S$.

$\boxed{\text{EX. 10}}$ Which of the following sets are spanning sets of $\mathbb{R}^3$?

a) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$

b) $\{(1,0,1)^T, (0,1,0)^T\}$

To determine whether a set of vectors in $\mathbb{R}^3$ spans the space $\mathbb{R}^3$, we need to consider a general vector $(a,b,c)^T$ in $\mathbb{R}^3$ and investigate if it can be written as a linear combination of the vectors in the set.

a) Need to determine if it is possible to find real constants $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = a \\ \alpha_1 + \alpha_2 = b \\ \alpha_1 = c \end{cases}$$

We can write the system of linear equations in matrix form:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Longrightarrow \det(A) = 1 \times 0 + 1 \times 0 + 1 \times (-1) = -1 \neq 0.$$

Since $\det(A) \neq 0$, there is a unique solution, which is $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix}$.

Hence, the three vectors span $\mathbb{R}^3$.

b) Any linear combination of $(1,0,1)^T$ and $(0,1,0)^T$ has the form:

$$\mathbf{x} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix}$$

Hence, $\text{span}((1,0,1)^T, (0,1,0)^T)$ contains only vectors whose 1st and 3rd entries are equal. Therefore the two vectors do not span $\mathbb{R}^3$. However, we can show that $\text{span}((1,0,1)^T, (0,1,0)^T)$ is a subspace of $\mathbb{R}^3$.

$\boxed{\text{Ex. 11}}$ Show that $1-x^2$ and $x+2$ and x^2 span P_3.

We need to show that any polynomial ax^2+bx+c in P_3 can be written as a linear combination of the three polynomials.

This means we want to show that it is possible to find $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha_1 (1-x^2) + \alpha_2 (x+2) + \alpha_3 x^2 = ax^2 + bx + c$$

$$\Rightarrow (-\alpha_1 + \alpha_3) x^2 + \alpha_2 x + \alpha_1 + 2\alpha_2 = ax^2 + bx + c$$

$$\Rightarrow \begin{cases} -\alpha_1 \quad + \alpha_3 = a \\ \alpha_2 = b \\ \alpha_1 + 2\alpha_2 = c \end{cases} \Rightarrow \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

A unique solution $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ exists iff $\det(A) \neq 0$.

$$\det(A) = (-1)(0) + (0)(0) + (1)(-1) = -1 \neq 0.$$

Hence the three polynomials span $\mathbb{R}^3$.