Examples of Ch. 2

Ex. 1 Find the determinant of the following matrices, and determine whether they are singular or nonsingular:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

1) $\det(A) = 1 \times 4 - 2 \times 3 = -2$

Since $\det(A) \neq 0$, A is nonsingular.

2) $\det(B) = 1 \times 4 - 2 \times 2 = 0$

Since $\det(B) = 0$, B is singular.

Ex. 2 Find the determinant of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

We will use the general formula based on the first row:

$$\det(A) = a_{11} \cdot m_{11} + a_{12} \cdot m_{12} + a_{13} \cdot m_{13}$$

$$M_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \Rightarrow m_{11} = (-1)^{1+1} \cdot \det(M_{11}) = 5 \times 9 - 8 \times 6 = -3$$

$$M_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} \Rightarrow m_{12} = (-1)^{1+2} \cdot \det(M_{12}) = -(4 \times 9 - 7 \times 6) = 6$$

$$M_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \Rightarrow m_{13} = (-1)^{1+3} \cdot \det(M_{13}) = 4 \times 8 - 7 \times 5 = -3$$

$$\Rightarrow \det(A) = 1 \times (-3) + 2 \times 6 + 3 \times (-3) = 0$$

Ex. 3 Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Verify that $\det(A) = \det(A^T)$.

1) $\det(A) = 1 \times 4 - 3 \times 2 = -2$

2) $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. $\det(A^T) = 1 \times 4 - 2 \times 3 = -2$

$\Rightarrow \det(A) = \det(A^T) = -2$

$\boxed{\text{EX. 4}}$ Find the determinant of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$.

Since A is a triangular matrix: $\det(A) = 1 \times 4 \times 6 = 24$.

$\boxed{\text{EX. 5}}$ Find the determinant of $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & -1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 5 & 7 & 0 & 2 \end{pmatrix}$.

Since the 1st and 3rd rows of A are identical, by Theorem $\boxed{T5}$ $\det(A) = 0$.

$\boxed{\text{EX. 6}}$ Find the determinant of $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 2 & 0 & 1 \end{pmatrix}$, where A is nonsingular.

Method 1 We use the general formula: $\det(A) = a_{11} \cdot m_{11} + a_{12} \cdot m_{12} + a_{13} \cdot m_{13}$

$M_{11} = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} \Rightarrow m_{11} = (-1)^{1+1} \det(M_{11}) = 3 \times 1 - 0 \times 5 = 3$

$M_{12} = \begin{pmatrix} -1 & 5 \\ 2 & 1 \end{pmatrix} \Rightarrow m_{12} = (-1)^{1+2} \det(M_{12}) = -((-1) \times 1 - 2 \times 5) = 11$

$M_{13} = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \Rightarrow m_{13} = (-1)^{1+3} \det(M_{13}) = (-1) \times 0 - 2 \times 3 = -6$

$\Rightarrow \det(A) = 1 \times 3 + 2 \times 11 + 3 \times (-6) = 7$

Method 2 Since we know that A is nonsingular, we reduce A to a triangular matrix T using only row operations of type I and III:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow[\text{on row 2}]{\text{operation type III}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow[\text{on row 3}]{\text{operation type III}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & -4 & -5 \end{pmatrix} \xrightarrow[\text{on row 3}]{\text{operation type III}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & \frac{7}{5} \end{pmatrix} = T$$

Since determinant of any elementary matrix of type III is one, and since we only used operation type III to obtain T, $\det(A) = \det(T) = 1 \times 5 \times \frac{7}{5} = 7$.

$\boxed{\text{EX. 7}}$ Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find A^{-1} using adj A.

We first find the cofactors of the entries of matrix A:

$m_{11} = (-1)^{1+1} \times 4 = 4$

$m_{12} = (-1)^{1+2} \times 3 = -3 \quad \Rightarrow \operatorname{adj} A = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$m_{21} = (-1)^{2+1} \times 2 = -2$

$m_{22} = (-1)^{2+2} \times 1 = 1$

$\det(A) = 1 \times 4 - 2 \times 3 = -2 \quad \Rightarrow \quad A^{-1} = \frac{\operatorname{adj} A}{\det(A)} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$